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ROYAL ASTRONOMICAL SOCIETY,

CONTAINING

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REPORTS OF THE PROCEEDINGS

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ROYAL ASTRONOMICAL SOCIETY.

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Major E. H. HILLS, C.M.G., F.R.S., PRESIDENT, in the Chair.

Nicolas Liapin, University Observatory, Odessa, Russia, was balloted for and duly elected a Fellow of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Stanley Herbert Buckland, Walsingham House, Whiteley Road, Upper Norwood, Surrey (proposed by Rev. T. E. R. Phillips);

Rev. Charles Evans, Elm House, Brenchley, Kent (proposed by A. M. Newbegin);

William Ernest Hartley, M.A., Chief Assistant, Cambridge Observatory (proposed by H. F. Newall);

Capt. Ronald L. Jones, Box 230, British Post Office, Shanghai, China (proposed by Rev. A. L. Cortie);

Capt. H. G. Lyons, R.E., F.R.S., 5 Heathview Gardens, Roehampton, S.W. (proposed by David Gill);

Rev. Edward O'Connor, S.J., M.A., Stonyhurst College, Blackburn, Lancashire (proposed by H. H. Turner); and

Henry Spencer Toy, B.Sc., Ashville College, Harrogate, and Helston, Cornwall (proposed by A. W. Porter).

One hundred and seventy presents were announced as having been received since the last Ordinary meeting, including, amongst others:—

Sir David Gill: History and description of the Royal Observatory, Cape of Good Hope, presented by the Astronomer Royal. Hamburger Sternwarte: Totale Sonnenfinsternis 1905 Aug. 30, presented by the Observatory. Jarry-Desloges: Observations des surfaces planétaires, presented by M. Jarry-Desloges. Memoirs of the British Astronomical Association: Light Curves of 21 Variable stars, presented by Col. Markwick and Mr. C. L. Brook. S. W. Burnham: Measures of proper motion stars, presented by the Carnegie Institution. Perth Observatory, Western Australia: Astrographic Catalogue, vols. ii. iii., presented by the Observatory. Astrographic Chart: Eighty-six charts, from Algiers and Paris Observatories, the Specola Vaticana, Rome, San Fernando and Tacubaya Observatories, and the Royal Observatory, Uccle, Belgium.

Note on a Method of Balancing Dome Shutters.

By W. H. Maw, LL.D.

For many years past the form of shutter-opening adopted for observatory domes of moderate or large size has generally been a segmental one, this opening being covered by a shutter (or pair of shutters) moving laterally. This arrangement is a convenient one in many respects, and if the shutters are properly designed and mounted, they can be operated easily.

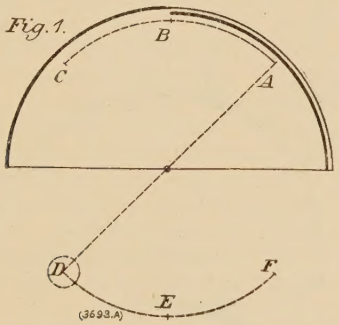
In the case of smaller domes—of, say, 20 feet diameter and under—the rectangular form of opening, fitted with a rising shutter, has certain advantages, and the reason why its use has been chiefly confined to quite small domes appears to consist in the difficulty in controlling the movement of the shutter, so that it may be readily used with any amount of opening desired. The ordinary unbalanced lifting shutter has of course only three positions in which it will remain without being secured, these being: completely closed, half open, and completely open; and of these three positions, the central one is the only one in which the shutter is in balance and can be moved readily. During the first part of the operation of opening the shutter—namely, from the closed position to half open—the shutter has to be lifted; while during the last period of the movement—from half open to completely open—it has to be prevented from overrunning, and in the case of a shutter of any but very small size these conditions involve considerable inconvenience.

Seventeen years ago—in 1896—I erected an observatory with an 18-foot dome at Outwood, Surrey, and decided to adopt the rectangular form of shutter-opening fitted with a rising shutter, this shutter, however, being so constructed as to be in complete

balance at all points of its path. The arrangement devised to secure this end has proved entirely satisfactory, and hence it has been thought that a brief note on it might be of some interest.

The principle on which the balancing is arranged may be explained by reference to diagram fig 1. In this diagram, A denotes approximately the position of the centre of gravity of a rising shutter in its closed position, and as the shutter is opened the centre of gravity will move successively to the positions B and C. Now it is evident that if it were possible to carry a lever across the dome, this lever being pivoted at its centre and having its upper end connected with the shutter and its lower end provided with a suitable balance weight, the shutter could be completely balanced and would remain indifferently in any position. But such an arrangement is of course impracticable, and the problem to be solved was to find a mechanical equivalent for this arrangement.

And here, I may say, that some years after my dome had been

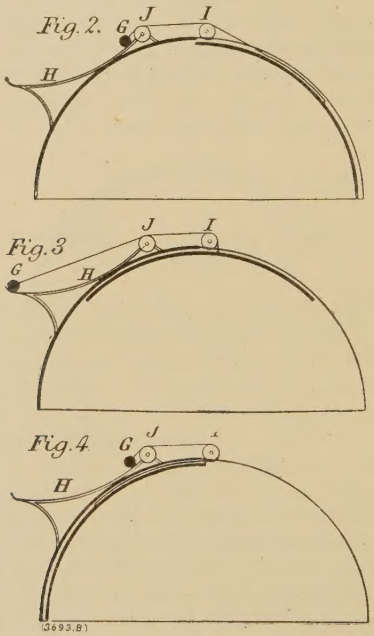


erected, I learned that there existed at Harvard College Observatory a small dome having a rising shutter balanced on the principle just explained. In this case the shutter travelled outside the dome casing, and the balancing lever had an arched form—like the handle of a pail,—the crown of the arch being connected to the shutter and the ends of the arch being pivoted on the base of the dome one on each side, and being extended downwards to carry balance weights. This arrangement was no doubt effective, but it would be somewhat cumbrous in any but a very small dome and would have other disadvantages. Whether or not this dome is now in use at Harvard, I do not know.

To return to fig. 1. In the arrangement there shown the balance weight would move in the arc D, E, F, and it is quite evident that if the lever across the dome was done away with and the shutter connected by some other means to a balance weight moving in a curved path D, E, F, so that the movements of the shutter and balance weight were coincident, the desired end would be attained. This would mean that as the shutter was opened the balance weight would first be lowered from D to E, and subsequently raised from

E to F. Now it is quite easy to couple the shutter and balance weights by a flexible connection which will transmit tension, but much more difficult to devise a connection which will resist compression and at the same time satisfy the other requirements existing in this particular case. It occurred to me, however, that as the two halves of the arc D, E, F were symmetrical, the proper thing to do instead of lowering the weight from D to E, and pushing it up from E to F, was to lower it to E and then pull it back again to D, the connection between the shutter and weight being thus always in tension.

Acting on this idea, I devised the arrangement which is shown



diagrammatically in figs. 2, 3 and 4, which respectively show the shutter closed, half open, and fully open. In this arrangement the balance weight G rolls on a pair of curved angle iron guides, H, mounted on the dome on the side opposite the shutter-opening. The balance weight is connected by a pair of light chains to the shutter at the centre of its length, each chain passing over a pair of guide pulleys I and J, placed as shown.

Fig. 2 shows the positions of the parts when the shutter is closed, the balance weight G being then at the top of its guides. In fig. 3 the shutter is half open, the balance weight being then at the bottom of the guides and exerting no pull on the connecting chains. The further opening motion of the shutter beyond the position shown in fig. 3 causes the connecting chains to pass *under*

the pulleys I, thus hauling the balance weight up the guides until the shutter is fully open, the parts then having the positions shown in fig. 4.

The shutter is opened and closed by cords running over guide rollers which keep the cords close to the inner side of the dome casing. The shutter being completely balanced will of course stay in any position, and the whole arrangement has been found to work well and to be very convenient.

Photographs showing exterior and interior views of the dome are appended.

The Dynamics of a Globular Stellar System. By A. S. Eddington, M.A., M.Sc., Plumian Professor.

The force which at any moment acts on a star may be regarded as made up of two parts, viz. the central attraction of the whole stellar system, and the force due to the chance arrangement of the stars in the immediate neighbourhood. The appropriate dynamical theory depends on the relative importance of these two components. By an argument from the existence of moving clusters of stars which, though widely separated in space, have preserved closely equal and parallel motions for a vast period of time, I have been led to the conclusion that the effect of the stars in the immediate neighbourhood is of negligible amount. This appears to be confirmed by direct numerical estimates of its probable value. The discussions in this paper relate therefore to stars moving under the general attraction of the system; there will be no chance deflections or encounters, and consequently the theory differs fundamentally from the dynamical theory of gases, to which some resemblance might at first sight have been expected.

The actual discontinuous field of gravitational force, due to a number of point-centres (stars), is thus replaced by a continuous field due to a smoothed distribution of density. It is necessary to form a rough idea of this density. We know of about 20 stars contained in a sphere whose limit corresponds to a parallax of $0''{\cdot}2$. Doubtless there are some others yet to be found; and there may be almost any number of dark stars. Although many of the 20 stars are much fainter than the sun, their masses are probably not greatly less. We shall probably be considerably underestimating the density if we take the matter in this sphere as ten times the sun's mass. If the universe were a globular system of this density, each star would describe an elliptic orbit about the centre of the system in 300,000,000 years—a period less than current estimates of the age of the earth. The period is independent of the size of the orbit or of the system. It seems probable then that a mature star will have made at least one, and probably many journeys round its orbit since it came into existence. We are therefore

justified in thinking of the stellar orbits as paths which have been actually traversed and not as mere theoretical curves.

If we wrote down at random an arbitrary distribution of motions in a stellar system, we should generally find on tracing backwards the past positions of the stars, that the distribution of stellar density became irregular and fantastic, such as we must reject as very unlikely to have occurred in the past history of the universe. By this kind of criterion certain velocity-distributions can be rejected, and others selected as possible—that is not leading to any inconsistency or improbability. Amongst the selected possible distributions, there is one important class, namely, those which correspond to a steady state of the stellar system. These clearly can lead to no inconsistency; for the past arrangement is the same as the present. I do not know that any evidence can be adduced to show that the stellar system is in a stationary or quasi-stationary state; but it seems to be the proper starting point in the study of a new kind of dynamical system; and it is entirely in accordance with the precedents of hydrodynamics and molecular dynamics to consider the possible steady states of motion.

The present paper hardly goes beyond the elements of the subject; the bearing of these simpler systems on the actual problem of stellar motions may be compared to the bearing of the study of periodic orbits on the actual planetary movements. I have, however, kept in view two main practical objects: (1) to examine whether these systems lead to a distribution of velocities approximating in any way to Maxwell's law, and (2) to examine Professor Turner's suggestion * that the two streams may be due to a preponderating radial motion of the stars, and in particular whether his hypothesis involves too great a congestion of stars at the centre of the system.

From a recent conversation with Professor Schwarzschild I learn that he has been engaged on the same problem. We appear to have followed the same steps, making the same hypothesis that the chance encounters can be neglected, and selecting the stationary states as the important ones for study. Even in such details as the solution by the use of Abel's integral equation, there was a similarity. I understand, however, that he has mainly studied a spheroidal system, whereas I have confined attention to globular systems, so that, happily, the work has not been duplicated.

It may be convenient to summarise the hypotheses and limitations of this work:—

(1) The stars are supposed to move under the central attraction of the whole system, undisturbed by their immediate neighbours.

(2) A search is made for distributions of velocity which yield a steady state for the whole system.

(3) The system is taken to be globular (*i.e.* to have spherical symmetry); the density is not necessarily uniform.

* *M. N.*, vol. lxxii. p. 387.

Let r be the distance of a star from the centre of the system.
u its radial velocity.
v its transverse velocity.
ϕ the gravitational potential at r.
σ the density at r.
Let the suffix o refer to the apcentron of the star's orbit.
The equations of energy and angular momentum give

$$\tfrac{1}{2}(u^2+v^2)-\phi=\tfrac{1}{2}v_0^2-\phi_0 \quad . \quad . \quad . \quad (1)$$

$$rv=r_0v_0 \quad . \quad . \quad . \quad . \quad (2)$$

Thus
$$u^2=2(\phi-\phi_0)-v_0^2\left(\frac{r_0^2}{r^2}-1\right) \quad . \quad . \quad . \quad (3)$$

or
$$u=\frac{\sqrt{(r_0^2-r^2)}}{r}\cdot\sqrt{(\gamma^2-v_0^2)} \quad . \quad . \quad . \quad (4)$$

where
$$\gamma=r\sqrt{\frac{2(\phi-\phi_0)}{r_0^2-r^2}} \quad . \quad . \quad . \quad . \quad (5)$$

It is clear that v_0 cannot be greater than γ; or, rather, that stars for which $v_0>\gamma$ remain outside the point r considered.

In a time-interval δt a certain number of stars will arrive at apcentron. Consider those whose apcentron distance is between r_0 and r_0+dr_0, and whose velocity at apcentron is between v_0 and v_0+dv_0. The number of these will be a function of r_0 and v_0; we shall write it in the form

$$a(r_0)dr_0\cdot f(v_0)dv_0\delta t.$$

Here $a(r_0)$ is a function of r_0 only; but we reserve the right of making $f(v_0)$ involve r_0 as well as v_0, though we usually make it a function of v_0 only. Any one of these stars will, during the next circuit, be within the shell r to $r+dr$ for a time equal to

$$\frac{2dr}{u}.$$

The factor 2 allows for going and returning.

The volume of the shell is $4\pi r^2dr$.

Thus, taking the time-average, the number of stars of this character per unit volume will be

$$d^2\sigma=\frac{a(r_0)dr_0\cdot f(v_0)dv_0}{2\pi r^2\cdot u} \quad . \quad . \quad . \quad (6)$$

I use the symbol $d^2\sigma$ for the part of σ corresponding to a double differential, in this case dr_0dv_0. I call this an *element of density*. After one integration $d\sigma$, called here the *partial density*, is obtained. A second integration gives σ.

By (4)

$$d^2\sigma=\frac{r}{\sqrt{(r_0^2-r^2)}}\cdot\frac{a(r_0)dr_0}{2\pi r^2}\cdot\frac{f(v_0)dv_0}{\sqrt{(\gamma^2-v_0^2)}} \quad . \quad . \quad (7)$$

We shall proceed to consider three special forms of $f(v_0)$.

Case (1). If $f(v_0) = \text{constant} = c$ (up to the limiting velocity γ), we have, integrating from $v_0 = 0$ to γ, the partial density,

$$d\sigma = \frac{r}{\sqrt{(r_0^2 - r^2)}} \cdot \frac{a(r_0)dr_0}{2\pi r^2} \cdot c\frac{\pi}{2},$$

$$= \frac{c}{4} \frac{a(r_0)dr_0}{r\sqrt{(r_0^2 - r^2)}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Case (2). If $f(v_0) = cv_0$,

$$d\sigma = \frac{r}{\sqrt{(r_0^2 - r^2)}} \cdot \frac{a(r_0)dr_0}{2\pi r^2} \cdot c\gamma,$$

$$= \frac{c}{2\pi} a(r_0)dr_0 \frac{\sqrt{2(\phi - \phi_0)}}{r_0^2 - r^2} \quad . \quad . \quad . \quad (9)$$

Case (3). If $f(v_0) = cv_0^2$,

$$d\sigma = \frac{c}{8} a(r_0)dr_0 \frac{2(\phi - \phi_0)r}{(r_0^2 - r^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad (10)$$

The result of the first assumption $f(v_0) = \text{constant}$ is remarkable, because it does not contain ϕ. The partial density due to stars with apcentron distance r_0 is independent of the total density. It is not easy to see the physical meaning or to give any reason for this curious property. Its use is, however, limited by the fact that the resulting value of $d\sigma$ contains a factor $\frac{1}{r}$, which must persist in σ. Thus it leads to systems with density infinite at the centre of order $\frac{1}{r}$ at least. Although this infinite density does not involve an infinite number of stars near the origin, and thus is not unreasonable, we naturally pay more attention to distributions in which the density runs up less sharply.

We now consider some definite laws of density.

I. Uniform density, with a sharp boundary at $r = R$.

In this case

$$2(\phi - \phi_0) = \tfrac{4}{3}\pi\sigma(r_0^2 - r^2),$$

and

$$\gamma = r\sqrt{\tfrac{4}{3}\pi\sigma} \quad . \quad . \quad . \quad . \quad (11)$$

As pointed out above, it is no use trying $f(v_0) = c$ with this law of density.

(*a*) If $f(v_0) = cv_0$,

$$d\sigma = \frac{c}{2\pi}\sqrt{\tfrac{4}{3}\pi\sigma} \cdot \frac{a(r_0)dr_0}{\sqrt{(r_0^2 - r^2)}}.$$

Thus

$$\sigma = \frac{c}{2\pi}\sqrt{\tfrac{4}{3}\pi\sigma} \cdot \int_r^R \frac{a(r_0)dr_0}{\sqrt{(r_0^2 - r^2)}}.$$

This is an integral equation to determine $a(r_0)$. It can be reduced to Abel's integral equation and the solution is found to be

$$a(r_0) = \frac{r_0}{\sqrt{(R^2 - r_0^2)}} \quad . \quad . \quad . \quad . \quad (12)$$

(*b*) $f(v_0) = cv_0^2$ leads similarly to

$$a(r_0) = \frac{1}{r_0\sqrt{(R^2 - r_0^2)}} \quad . \quad . \quad . \quad (13)$$

(The solutions (12) and (13) are easily verified as follows :— We have to verify

(*a*) $$\sigma = \frac{c}{2\pi}\sqrt{\tfrac{4}{3}\pi\sigma}\int_r^R \frac{r_0 dr_0}{\sqrt{(R^2 - r_0^2)}\sqrt{(r_0^2 - r^2)}},$$

(*b*) $$\sigma = \frac{c}{8}\tfrac{4}{3}\pi\sigma\int_r^R \frac{r dr_0}{r_0\sqrt{(R^2 - r_0^2)}\sqrt{(r_0^2 - r^2)}},$$

setting $r_0^2 = R^2\sin^2\theta + r^2\cos^2\theta,$

$$\frac{r_0 dr_0}{\sqrt{(R^2 - r_0^2)}\sqrt{(r_0^2 - r^2)}} = d\theta,$$

these reduce to (*a*) $\sigma = \frac{c}{4}\sqrt{\tfrac{4}{3}\pi\sigma}$

(*b*) $\sigma = \frac{c}{R}\cdot\frac{\pi}{16}\cdot\tfrac{4}{3}\pi\sigma$ $\Bigg\}$ . . (14)

which are satisfied by a suitable choice of the constant c in each case.)

Having found these two solutions of the problem proposed, we shall now consider the solution (*a*) further.

By (6) and (12), the element of density is

$$d^2\sigma = \frac{c}{2\pi}\frac{v_0 dv_0}{r^2 u}\frac{r_0 dr_0}{\sqrt{(R^2 - r_0^2)}} \quad . \quad . \quad . \quad (15)$$

Also by (4) and (11)

$$u^2 = \frac{r_0^2 - r^2}{r^2}\left(\tfrac{4}{3}\pi\sigma r^2 - v_0^2\right).$$

Let us change the variables from r_0, v_0, to u, v_0. Then, keeping v_0 constant,

$$u du = \frac{r_0 dr_0}{r^2}\left(\tfrac{4}{3}\pi\sigma r^2 - v_0^2\right),$$

and
$$R^2 - r_0^2 = (R^2 - r^2) - (r_0^2 - r^2)$$
$$= (R^2 - r^2) - \frac{r^2 u^2}{\tfrac{4}{3}\pi\sigma r^2 - v_0^2}.$$

Thus

$$d^2\sigma = \frac{c}{2}\frac{du}{\sqrt{(R^2 - r^2)}}\cdot\frac{v_0 dv_0}{\sqrt{(\tfrac{4}{3}\pi\sigma r^2 - v_0^2)}\cdot\sqrt{\left(\tfrac{4}{3}\pi\sigma r^2 - \frac{r^2 u^2}{R^2 - r^2} - v_0^2\right)}}.$$

Now perform the integration with respect to v_0. The limits are clearly from 0 to

$$\sqrt{\left(\tfrac{4}{3}\pi\sigma r^2 - \frac{r^2u^2}{R^2-r^2}\right)}.$$

The result is found to be

$$d\sigma = \frac{c}{2\pi}\frac{du}{\sqrt{(R^2-r^2)}}\cosh^{-1}\frac{\sqrt{\tfrac{4}{3}\pi\sigma(R^2-r^2)}}{u}. \quad . \quad (16)$$

This partial density corresponding to the radial velocities u to $u+du$ is what is usually called the frequency of u.

Setting $a = \sqrt{\tfrac{4}{3}\pi\sigma(R^2-r^2)}$, we see that the frequency of u is proportional to

$$\operatorname{sech}^{-1}\frac{u}{a}.$$

This is infinite for a zero velocity and decreases to zero for the limiting velocity $u=a$. u cannot exceed a.

The mean radial velocity, irrespective of sign (at the distance r from the centre), is

$$\int_0^a u\operatorname{sech}^{-1}\frac{u}{a}du \div \int_0^a \operatorname{sech}^{-1}\frac{u}{a}du$$

$$= \tfrac{1}{2}a^2 \div \frac{\pi}{2}a = \frac{a}{\pi} = \frac{1}{\pi}\sqrt{\tfrac{4}{3}\pi\sigma(R^2-r^2)} \quad . \quad . \quad (17)$$

For the transverse velocities we transform to variables v and v_0,

$$r_0dr_0 = \frac{r^2}{v_0^2}vdv,$$

and (15) becomes

$$d^2\sigma = \frac{c}{2\pi R}vdv\frac{v_0dv_0}{\sqrt{(\tfrac{4}{3}\pi\sigma r^2 - v_0^2)}\sqrt{(v^2-v_0^2)}\sqrt{\left(v_0^2 - v^2\frac{r^2}{R^2}\right)}}.$$

Setting now

$$t = v_0^2$$
$$d = \tfrac{4}{3}\pi\sigma r^2$$
$$e = v^2$$
$$f = v^2\frac{r^2}{R^2}$$

the partial density is

$$d\sigma = \frac{cvdv}{4\pi R}\int_f \frac{dt}{\sqrt{(d-t)(e-t)(t-f)}}.$$

The upper limit being d or e, whichever is the least.

This reduces to

$$d\sigma = \frac{c}{2\pi R}vdv\frac{1}{\sqrt{(d-f)}}F^1\left\{\sqrt{\frac{e-f}{d-f}}\right\} \text{ if } v^2 < \tfrac{4}{3}\pi\sigma r^2$$

$$= \frac{c}{2\pi R}vdv\frac{1}{\sqrt{(e-f)}}F^1\left\{\sqrt{\frac{d-f}{e-f}}\right\} \text{ if } v^2 > \tfrac{4}{3}\pi\sigma r^2.$$

Here F^1 is the complete elliptic integral.

These expressions are equivalent to

$$d\sigma = \frac{c}{2\pi r} v dv \frac{1}{\sqrt{(\frac{4}{3}\pi\sigma \mathrm{R}^2 - v^2)}} \mathrm{F}^1 \left\{ \frac{\sqrt{(\mathrm{R}^2 - r^2)}}{r} \cdot \frac{v}{\sqrt{(\frac{4}{3}\pi\sigma \mathrm{R}^2 - v^2)}} \right\}$$

and

$$d\sigma = \frac{c}{2\pi \sqrt{(\mathrm{R}^2 - r^2)}} \cdot dv \cdot \mathrm{F}^1 \left\{ \frac{r}{\sqrt{(\mathrm{R}^2 - r^2)}} \cdot \frac{\sqrt{(\frac{4}{3}\pi\sigma \mathrm{R}^2 - v^2)}}{v} \right\} \quad (18)$$

The frequency of a resultant transverse velocity v is equal to the coefficient of dv; but the combined frequency of the two components, which make up a resultant velocity v, is proportional to the coefficient of vdv.

This latter coefficient increases very slowly from a constant at $v = 0$, until v approaches $r\sqrt{\frac{4}{3}\pi\sigma}$, when it rapidly becomes infinite; it then decreases again, at first rapidly, then rather slowly to another constant. Except when v is nearly equal to $r\sqrt{\frac{4}{3}\pi\sigma}$ the elliptic integral does not differ much from unity.

The mean transverse motion, irrespective of sign, can be determined as follows:—

By (6) $$d^2\sigma = \frac{a(r_0) dr_0 f(v_0) dv_0}{2\pi r^2 u},$$

and by (2) and (4) $$\frac{v}{u} = \frac{v_0 r_0}{r} \cdot \frac{r}{\sqrt{(r_0^2 - r^2)}} \cdot \frac{1}{\sqrt{(\gamma^2 - v_0^2)}} \quad . \quad (19)$$

Thus $$v d^2\sigma = \frac{r_0}{\sqrt{(r_0^2 - r^2)}} \cdot \frac{a(r_0) dr_0}{2\pi r^2} \cdot \frac{v_0 f(v_0) dv_0}{\sqrt{(\gamma^2 - v_0^2)}}, \quad . \quad (20)$$

a perfectly general expression.

Substituting $f(v_0) = cv_0$, we see that the integration with respect to v_0 is identical with that of (7), case (3). Thus by (10)

$$\int v d^2\sigma = \frac{c}{8} a(r_0) dr_0 \cdot \frac{2(\phi - \phi_0) r_0}{(r_0^2 - r^2)^{\frac{3}{2}}},$$

and $$\iint v d^2\sigma = \frac{c}{8} \cdot \int_r^{\mathrm{R}} \frac{r_0^2 dr_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}} \cdot \frac{\frac{4}{3}\pi\sigma}{\sqrt{(r_0^2 - r^2)}},$$

or writing $$r_0^2 = \mathrm{R}^2 \sin^2\theta + r^2 \cos^2\theta,$$

$$\iint v d\sigma = \frac{c}{8} \cdot \tfrac{4}{3}\pi\sigma \int_0^{\frac{\pi}{2}} \sqrt{(\mathrm{R}^2 \sin^2\theta + r^2 \cos^2\theta)} d\theta.$$

Also $$\iint d\sigma = \sigma = \frac{c}{4} \sqrt{\tfrac{4}{3}\pi\sigma}, \text{ by (14).}$$

Thus mean value of v

$$= \tfrac{1}{2} \sqrt{\tfrac{4}{3}\pi\sigma} \int_0^{\frac{\pi}{2}} \sqrt{(\mathrm{R}^2 \sin^2\theta + r^2 \cos^2\theta)} d\theta$$

$$= \tfrac{1}{2} \sqrt{\tfrac{4}{3}\pi\sigma} \cdot \mathrm{R} \cdot \mathrm{E}^1 \left\{ \frac{\sqrt{(\mathrm{R}^2 - r^2)}}{\mathrm{R}} \right\}.$$

The mean value of a one-dimensional component of v is $\frac{2}{\pi}$ times this,

$$= \frac{1}{\pi} \sqrt{\tfrac{4}{3}\pi\sigma} \cdot \mathrm{R} \cdot \mathrm{E}^1 \left\{ \frac{\sqrt{(\mathrm{R}^2 - r^2)}}{\mathrm{R}} \right\} . \quad . \quad (21)$$

As r goes from 0 to R, E^1 changes from 1 to $\frac{\pi}{2}$. Comparing (21) and (17) we see that the mean values of u and of a component of v are equal for $r = 0$, and that the former diminishes and the latter increases as r passes from 0 to R. The motion is thus *not* preferentially radial, but except at the centre it is preferentially transverse, corresponding to an oblate velocity-ellipsoid. Thus this system does not provide an explanation of star-streaming

II. Density varies as $\frac{1}{r}$, with an abrupt boundary at $r = \mathrm{R}$.

Although the destiny at the centre is infinite, the number of stars is finite. For this case a solution can be obtained when $f(v_0) = \text{constant} = c$.

By (8)
$$d\sigma = \frac{c}{4} \frac{a(r_0) dr_0}{r\sqrt{(r_0^2 - r^2)}} .$$

Putting $\sigma = \frac{k}{r}$ we must have

$$k = \frac{c}{4} \int_r^{\mathrm{R}} \frac{a(r_0) dr_0}{\sqrt{(r_0^2 - r^2)}} .$$

The solution is
$$a(r_0) = \frac{r_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}}$$

and c is given by

$$k = \frac{c}{4} \cdot \frac{\pi}{2} .$$

Without working out the actual distribution of velocities, we can compare the mean velocity components u and v as follows:—

$$u d^2\sigma = \frac{c}{2\pi r^2} \cdot \frac{r_0 dr_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}} \cdot dv_0,$$

and by (20),

$$v d^2\sigma = \frac{c}{2\pi r^2} \cdot \frac{r_0 dr_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}} \cdot \frac{r_0}{\sqrt{(r_0^2 - r^2)}} \cdot \frac{v_0 dv_0}{\sqrt{(\gamma^2 - v_0^2)}} .$$

Hence performing the v_0 integration from 0 to γ.

$$\int u d^2\sigma = \frac{c}{2\pi r^2} \frac{r_0 dr_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}} \cdot \gamma,$$

$$\int v d^2\sigma = \frac{c}{2\pi r^2} \cdot \frac{r_0 dr_0}{\sqrt{(\mathrm{R}^2 - r_0^2)}} \cdot \frac{r_0}{\sqrt{(r_0^2 - r^2)}} \cdot \gamma.$$

These reduce to the same expression when r is small; but otherwise, for each element, the second expression is greater than the

first. Thus, the mean value of v is always greater than that of u, except at the centre of the system, where they approach equality.

At the centre the mean value of a one-dimensional component of $v = \frac{2}{\pi} \times$ mean value of u.

There is thus preferential motion in the radial direction at the centre, the Schwarzschild ellipsoid having axes in the ratio $1 : \frac{2}{\pi}$, *i.e.* $1 : \cdot 64$. Away from the centre the preferential motion diminishes. The actually observed elongation of the ellipsoid is rather greater than this.

III. Law of density $\sigma = \frac{1}{(1+r^2)^{\frac{5}{2}}}$.

This is the law which Professor H. C. Plummer has found to represent the density in certain globular clusters. It is a convenient type of a distribution with a large central part of nearly uniform density, thinning out rapidly at the edges.

With this law $\phi = \frac{\frac{4}{3}\pi}{(1+r^2)^{\frac{1}{2}}}$.

Take $f(v_0) = cv_0$;

then $d\sigma = \frac{c}{2\pi} a(r_0) dr_0 \frac{\sqrt{2(\phi - \phi_0)}}{r_0^2 - r^2}$ (equation (9))

Setting $r = \tan\theta,\ r_0 = \tan\theta_0,$

so that $\sigma = \cos^5\theta,\ \phi = \frac{4}{3}\pi \cos\theta,$

we have $\cos^3\theta = \frac{c}{2\pi}\int_{\theta}^{\frac{\pi}{2}} a(\tan\theta_0) d\theta_0 \cdot \frac{\sqrt{\frac{4}{3}\pi(\cos\theta - \cos\theta_0)}}{\cos^2\theta - \cos^2\theta_0}$.

The solution is $a(\tan\theta_0) = \cos^{\frac{7}{2}}\theta_0 \sin\theta_0$,

for then $\cos^3\theta = \frac{c}{2\pi}\int_0^{\cos\theta} \cos^{\frac{7}{2}}\theta_0 d(\cos\theta_0) \frac{\sqrt{\frac{4}{3}\pi(\cos\theta - \cos\theta_0)}}{\cos^2\theta - \cos^2\theta_0}$,

or putting $t = \frac{\cos\theta_0}{\cos\theta}$

$$1 = \frac{c\sqrt{\frac{4}{3}\pi}}{2\pi}\int_0^1 t^{\frac{1}{2}} \frac{\sqrt{(1-t)}}{1-t^2} dt.$$

The integral is merely a numerical factor determining c, which could be evaluated if necessary.

In terms of r_0 the solution becomes

$$a(r_0) = \frac{\tan\theta_0}{\sec^{\frac{9}{2}}\theta_0}$$

$$= \frac{r_0}{(1+r_0^2)^{\frac{9}{4}}}.$$

I have not investigated the corresponding laws of velocity.

It will be convenient to summarise our work up to this point. We have found four examples of stellar systems in a steady state, viz.:—

No.	Law of Density.	Boundary.	$f(v_0)$.	$a(r_0)$.
1	constant	R	cv_0	$r_0(R^2-r_0^2)^{-\frac{1}{2}}$
2	constant	R	cv_0^2	$r_0^{-1}(R^2-r_0^2)^{-\frac{1}{2}}$
3	r^{-1}	R	c	$r_0(R^2-r_0^2)^{-\frac{1}{2}}$
4	$(1+r^2)^{-\frac{5}{2}}$	unlimited	cv_0	$r_0(1+r_0^2)^{-\frac{3}{4}}$

Of these we found that No. 1 gave no preferential motion in the radial direction, and No. 3 gave a preferential motion amounting to at most 1 : ·64. Further, we worked out in detail the velocity distribution corresponding to No. 1.

I shall now prove a rather general theorem.

If the law of aphelion velocities is such that $f(v_0)$ increases *with v_0 up to the limiting velocity for that distance, then under no circumstances can the preferential motion in the radial direction be greater than* $1 : \frac{2}{\pi}$.

By (6) and (20), which are both quite general formulæ,

$$ud^2\sigma = \frac{a(r_0)dr_0}{2\pi r^2} \cdot f(v_0)dv_0 \quad . \quad . \quad . \quad . \quad (22)$$

$$vd^2\sigma = \frac{a(r_0)dr_0}{2\pi r^2} \cdot \frac{r_0}{\sqrt{(r_0^2-r^2)}} \cdot \frac{f(v_0)v_0dv_0}{\sqrt{(\gamma^2-v_0^2)}} \quad . \quad (23)$$

Set $$\mathsf{U} = \int_0^\gamma f(v_0)dv_0 = \int_0^\gamma f(\gamma - v_0)dv_0,$$

$$\mathsf{V} = \int_0^\gamma \frac{f(v_0)v_0}{\sqrt{\gamma^2-v_0^2}}dv_0,$$

$$= -\int_0^\gamma f(v_0)d(\sqrt{(\gamma^2-v_0^2)}),$$

$$= \int_0^\gamma f(\sqrt{(\gamma^2-v_0^2)})dv_0.$$

But by the hypothesis—

since $$\sqrt{(\gamma^2-v_0^2)} > \gamma - v_0,$$
$$f(\sqrt{(\gamma^2-v_0^2)}) > f(\gamma - v_0),$$

therefore $$\mathsf{V} > \mathsf{U}.$$

Further, the other factor, by which (23) differs from (22), viz. $\frac{r_0}{\sqrt{(r_0^2-r^2)}}$, is always greater than unity except when $r=0$, when it is equal to unity.

Thus $$\iint vd^2\sigma > \iint ud^2\sigma,$$

and mean value of $v >$ mean value of u.

Finally, the mean value of a one-dimensional component of v : mean value of u in a ratio greater than $\frac{2}{\pi}$: 1.

A complementary theorem is as follows :—

If the law of apcentron velocities is such that $f(v_0)$ diminishes *with* v_0, *the density at the centre of the system will be infinite at least to the order* r^{-1}.

I have not been able to prove this quite rigorously, but it seems certain. If $f(v_0) = cv_0^{-\lambda}$,

by (7), $$d^2\sigma = \frac{r}{\sqrt{(r_0^2 - r^2)}} \frac{a(r_0)dr_0}{2\pi r^2} \frac{cv_0^{-\lambda}dv_0}{\sqrt{(\gamma^2 - v_0^2)}}.$$

By considering dimensions $$\int_0^{\gamma} \frac{v_0^{-\lambda}dv_0}{\sqrt{(\gamma^2 - v_0^2)}} \propto \gamma^{-\lambda}$$

$$\propto r^{-\lambda} \left\{ \frac{2(\phi - \phi_0)}{r_0^2 - r^2} \right\}^{-\frac{\lambda}{2}},$$

thus the partial density $d\sigma$ is of the form

$$\frac{a(r_0)dr_0}{\sqrt{(r_0^2 - r^2)}} \left\{ \frac{2(\phi - \phi_0)}{r_0^2 - r^2} \right\}^{-\frac{\lambda}{2}} \cdot \frac{1}{r^{1+\lambda}},$$

and it is clear that the factor $\frac{1}{r^{1+\lambda}}$ will persist in the next integration with respect to r_0.

Further, the stars which pass actually through the centre are those for which $v_0 = 0$; thus we need not suppose that $f(v_0) = cv_0^{-\lambda}$ throughout the range, but only that it starts from zero like $v_0^{-\lambda}$.

An evident corollary is that, to avoid an infinite density at the centre, $f(v_0)$ must at first increase at least as rapidly as v_0.

Between these two theorems it is clear that we should turn attention to forms of $f(v_0)$, which do not continuously increase or decrease, but increase to a maximum and then diminish. Any other possibility of obtaining adequate preferential motion without a great congestion at the centre is shut out. The forms that naturally suggest themselves are $f(v_0) = v_0 e^{-h^2 v_0^2}$, where h may be a function of r_0, and $f(v_0) = v_0(c^2 - v_0^2)^p$, where c may be a function of r_0. A tempting form is $v_0(\gamma^2 - v_0^2)^p$; but γ involves r as well as r_0, and the form is inadmissible.

It may be well to add a word of warning to anyone making further investigations. The determination of $a(r_0)$ by means of the integral equation is the main difficulty. We have been able, in certain cases, to draw conclusions as to the mean motions in the radial and transverse directions without using the value of $a(r_0)$. But we must always prove that there is a possible solution for $a(r_0)$ positive for all values of r_0, satisfying the law of $f(v_0)$ and the law of density that have been adopted. It is somewhat annoying to deduce a number of interesting properties of a system, which is afterwards found not to exist. In this respect I have to correct a remark made at the British Association,

1913. I stated that the law $f(v_0) = v_0 e^{-h^2 v_0^2}$ would yield a large preferential motion without any congestion at the centre. Through a mistaken generalisation, I had taken for granted that the system I was discussing was possible. I am now uncertain as to this. I think I have obtained a proof that uniform density is *impossible* with this law (if h is constant, which was the case considered); but I am not sure whether the density is necessarily infinite at the centre.

The extension to these more complex forms of $f(v_0)$ presents considerable difficulties, and it may be necessary to resort to approximate and numerical methods of computation of $a(r_0)$. The present paper is only an introduction to the subject, but it is hoped that it may call attention to a hitherto uninvestigated kind of dynamical system.

Further Remarks on the Expression of Sun-spot Periodicity as a Fourier Sequence. (Papers of the I.U.S.R. Computing Bureau, No. XII.) By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. In the last paper (*M.N.*, lxxiii. p. 714) I gave the terms of the Fourier sequence for the first 54 submultiples of 156 years (p. 722, *loc. cit.*), but those with large coefficients are chiefly in the neighbourhood of the well-known 11-year period.

In the present paper the general significance of these important terms is examined. It is found that the main features of the record can be represented by four periodic terms, of periods approximately $8\frac{1}{3}$, $10\frac{1}{4}$, $11\frac{2}{5}$, $14\frac{2}{3}$ years, but that the coefficients do not remain constant, and only that of the third (11·4-year) period is sensible at the present time.

This result is not very different from that obtained by Professor Schuster [*Phil. Trans.*, 206, A (1906)] for the first and last periods, though the 11-year periodicity has been divided into two. The value 11·4 years for the main period was foreshadowed in the last paper (see pp. 719 and 726), and accords with that found by A. A. Michelson (*Ap. J.*, xxxviii. p. 268), whose paper was received after the greater portion of this paper was written.

It was hoped that a careful scrutiny might reveal the present activity of one or more of the subsidiary periods, thus enabling a prediction to be made for the immediate future; but this hope has proved illusory. The scrutiny by the present method has, however, an interest of its own.

2. The interpretation of the figures will perhaps be more easily followed if one or two simple cases are first considered.

Taking the Fourier sequence either of sines or cosines, suppose we have two consecutive large terms. For simplicity

let all others be zero, and let the coefficients of these two terms be unity, and

(*a*) *of the same sign.*

Thus the pair of terms are

$$\sin q\theta + \sin (q+1)\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where q is an integer, and θ varies from 0 to 2π. We may write them in the form

$$(2\cos \tfrac{1}{2}\theta) \,.\, \sin (q+\tfrac{1}{2})\theta. \quad . \quad . \quad . \quad (2)$$

i.e. we have a term with intermediate period $2\pi/(q+\frac{1}{2})$, and a variable coefficient $(2\cos\frac{1}{2}\theta)$ which takes all values from 2 to -2, as θ changes from 0 to 2π. Now this is equivalent to saying that *within the limits of past experience* the coefficient of the combined term is as often negative as positive, and its mean value is zero. We can therefore regard this as a *spurious* periodicity; or we can regard it as a *double* periodicity, recurring to the form (1). But we cannot regard it as a single periodicity; as may be illustrated thus. We may take as a type of such a periodicity

$$\begin{aligned} &2\cos^2 \tfrac{1}{2}\theta \,.\, \sin q\theta \\ &= (1+\cos\theta)\sin q\theta \\ &= \tfrac{1}{2}\sin (q-1)\theta + \sin q\theta + \tfrac{1}{2}\sin (q+1)\theta, \end{aligned}$$

thus requiring *three* consecutive Fourier terms instead of *two*.

The *pair of consecutive terms of the same sign* seems to typify "interference." The characteristic of "interference" is the dying-out of the combined wave, with subsequent reappearance in the opposite phase.

3. But if the pair of consecutive terms be

(*b*) *of opposite signs*,

then they may be written in the simplest case

$$\sin q\theta - \sin (q+1)\theta = -(2\sin\tfrac{1}{2}\theta) \,.\, \cos (q+\tfrac{1}{2})\theta.$$

Now $2\sin\frac{1}{2}\theta$ remains of the same sign throughout past experience, which is represented by $\theta = 0$ to $\theta = 2\pi$. It is small at the beginning and end, and large in the middle: but it is of constant sign; and this is the best we can do with only this pair of terms towards representing a term of period $2\pi/(q+\frac{1}{2})$ with a constant coefficient. By adding other terms we can approximate rapidly to a more uniform coefficient, and with the complete (infinite) series we can fully represent this term, as shown in the last paper. Thus the next pair outside would be

$$\tfrac{1}{3}\sin (q-1)\theta - \tfrac{1}{3}\sin (q+2)\theta = -(\tfrac{2}{3}\sin\tfrac{3}{2}\theta)\cos (q+\tfrac{1}{2})\theta.$$

The coefficient of $\cos (q+\frac{1}{2})\theta$ made up from these four terms will thus have the following values at

θ	$=0°$	60°	120°	180°	240°	300°	360°
$-2\sin\frac{1}{2}\theta =$	0·00	−1·00	−1·74	−2·00	−1·74	−1·00	0·00
$-\frac{2}{3}\sin\frac{3}{2}\theta =$	0·00	−0·67	0·00	+0·67	0·00	−0·67	0·00
Sum	=0·00	−1·67	−1·74	−1·33	−1·74	−1·67	0·00

Without following the series further we can see the way in which it approximates to a uniform coefficient: and the first pair of terms lays a good foundation.

4. To sum up; wherever we find *changes of sign* in the Fourier sequence, either of sine or cosine, we may expect to be able to combine the terms into a simple and more or less permanent periodicity: but *continuities of sign* can only mean spurious periodicities, dual or multiple periodicities, or periodicities which die out and reappear.

5. Let us now recur to Table I. of the last paper, which gives the coefficients of the Fourier sequence for a period $2\pi/(q+x)$, where x has different fractional values, as follows:—

TABLE I.

Fourier Terms.

Value of x	$q-4$	$q-3$	$q-2$	$q-1$	q	$q+1$	$q+2$	$q+3$
0·00	0·00	0·00	0·00	0·00	1·00	0·00	0·00	0·00
0·17	+·04	+·05	+·07	+·14	+·96	−·19	−·09	−·06
0·33	+·06	+·08	+·12	+·21	+·83	−·41	−·17	−·10
0·50	+·07	+·09	+·13	+·21	+·64	−·64	−·21	−·13

Now let us form differences between these coefficients, as follows:—

TABLE II.

Differences.

Value of x	. . . . .	. . . . .	a	b	c	. . .	$a-b+c$
0·00	·00	·00	·00	+1·00	−1·00	0·00	0·00	2·00
0·17	+·01	+·02	+·07	+·82	−1·15	+·10	+·03	2·07
0·33	+·02	+·04	+·09	+·62	−1·24	+·24	+·07	2·10
0·50	+·02	+·04	+·08	+·43	−1·28	+·43	+·08	2·14

When there is a simple periodic term we thus get one of these differences (b) opposite in sign to all the rest: and the value of $(a-b+c)$, formed in the last column, is seen to be nearly constant. Thus a rough knowledge of the value of x will give us a fairly accurate value of the coefficient.

6. But if there is another sensible periodicity in the neighbourhood the value of $a-b+c$ will be affected. Let us form this expression for consecutive terms of the above table. We get

TABLE III.

Values of $(a-b+c)$.

x						
0·00	·00	+1·00	−2·00	+2·00	−1·00	·00
0·17	+·06	+·77	−1·90	+2·07	−1·22	+·09
0·33	+·07	+·57	−1·77	+2·10	−1·41	+·20
0·50	+·06	+·39	−1·63	+2·14	−1·63	+·39

Thus the existence of a single periodicity disturbs the values of the second difference denoted by $(a-b+c)$ over a considerable range of terms. We must then proceed by a series of approximations, removing the approximate effect of one term before attempting to determine the coefficient of one in the neighbourhood. But we may clearly use the values of $(a-b+c)$ as a final test that no large periodicity, or no serious correction to a simple periodicity, has been overlooked: for if that were the case we could scarcely fail to see the effects in large terms, and rapid alternations of sign, in the series for $(a-b+c)$.

7. Having thus indicated the process of determining such terms numerically, and a final test for their accuracy, it is perhaps scarcely necessary to give the process in detail. All the essential information is contained in the following two tables for S and C, including the material for further refinements if desired. There appear to be four large terms of simply-periodic character. The coefficients of S and C declare themselves independently, but the periods must, of course, be considered together; and when the indications of the two series differ it seems reasonable to assign weights in proportion to the size of the coefficients, which, as already remarked, are approximately determinable from a rough knowledge of the periods.

TABLE IV.

Four Principal Terms of S *Series.*

Divisor.	S.	Diff.	Divisors. 10·67.	13·67.	15·25.	18·67.	Residue.	Diff.	$a+c-b$.
8	+ 3·6		+ 0·7	+ 1·2	− 0·1	+ 0·2	+ 1·6		
		− 2·3						− 3·1	...
9	+ 1·3		+ 1·2	+ 1·5	− 0·2	+ 0·3	− 1·5		
		+ 0·5						− 1·6	− 0·6
10	+ 1·8		+ 3·0	+ 1·9	− 0·3	+ 0·3	− 3·1		
		− 7·3						+ 0·9	+ 3·0
11	− 5·5		− 5·9	+ 2·7	− 0·4	+ 0·3	− 2·2		
		+11·4						+ 5·5	− 2·6
12	+ 5·9		− 1·5	+ 4·3	− 0·6	+ 0·4	+ 3·3		
		+ 8·8						+ 2·0	+ 1·4
13	+14·7		− 0·8	+10·6	− 0·8	+ 0·4	+ 5·3		
		−34·3						− 2·1	+ 0·4
14	−19·6		− 0·6	−21·2	− 1·5	+ 0·5	+ 3·2		
		+ 6·4						− 3·7	+ 2·5
15	−13·2		− 0·5	− 5·3	− 7·6	+ 0·7	− 0·5		
		+13·6						+ 0·9	+ 1·3
16	+ 0·4		− 0·4	− 3·0	+ 2·5	+ 0·9	+ 0·4		
		+ 6·1						+ 5·9	− 3·5
17	+ 6·5		− 0·3	− 2·1	+ 1·1	+ 1·5	+ 6·3		
		+ 3·8						+ 1·5	+ 2·8
18	+10·3		− 0·2	− 1·7	+ 0·7	+ 3·7	+ 7·8		
		−12·6						− 1·6	− 0·6
19	− 2·3		− 0·2	− 1·4	+ 0·5	− 7·4	+ 6·2		
		+ 1·9						− 3·7	− 1·0
20	− 0·4		− 0·1	− 1·2	+ 0·3	− 1·9	+ 2·5		
		− 2·2						− 3·1	− 3·9
21	− 2·6		− 0·1	− 1·0	+ 0·2	− 1·1	− 0·6		
		− 2·7						− 3·3	+ 3·3
22	− 5·3		− 0·1	− 0·8	+ 0·2	− 0·7	− 3·9		
		+ 3·4						+ 3·1	− 4·8
23	− 1·9		0·0	− 0·6	+ 0·1	− 0·6	− 0·8		
		+ 1·8						+ 1·6	...
24	− 0·1		0·0	− 0·5	+ 0·1	− 0·5	+ 0·8		

TABLE V.

Four Principal Terms of C *Series.*

Divisor.	C.	Diff.	Divisors. 10·67.	13·67.	15·25.	18·67.	Residue.	Diff.	$a+c-b$.
8	+ 4·5		− 0·8	− 0·4	− 0·6	+ 0·2	+ 6·1		
		− 0·8						0·0	...
9	+ 3·7		− 1·3	− 0·6	− 0·7	+ 0·2	+ 6·1		
		− 9·6						− 7·2	+ 6·8
10	− 5·9		− 3·3	− 0·8	− 0·9	+ 0·2	− 1·1		
		+ 9·1						− 0·4	+ 0·8
11	+ 3·2		+ 6·6	− 1·1	− 1·1	+ 0·3	− 1·5		
		+ 1·6						+ 7·6	− 5·7
12	+ 4·8		+ 1·7	− 1·8	− 1·5	+ 0·3	+ 6·1		
		− 1·7						+ 2·3	+ 6·3
13	+ 3·1		+ 0·9	− 4·5	− 2·1	+ 0·4	+ 8·4		
		+12·7						+ 1·0	+ 2·5
14	+15·8		+ 0·7	+ 8·9	− 3·7	+ 0·5	+ 9·4		
		−20·6						+ 1·2	− 5·9
15	− 4·8		+ 0·5	+ 2·2	−18·7	+ 0·6	+10·6		
		+18·4						− 5·7	+ 0·5
16	+13·6		+ 0·4	+ 1·3	+ 6·2	+ 0·8	+ 4·9		
		− 9·9						− 6·4	− 2·6
17	+ 3·7		+ 0·3	+ 0·9	+ 2·7	+ 1·3	− 1·5		
		− 2·5						− 3·3	− 3·9
18	+ 1·2		+ 0·3	+ 0·7	+ 1·7	+ 3·3	− 4·8		
		−11·5						− 0·8	+ 1·0
19	−10·3		+ 0·2	+ 0·6	+ 1·2	− 6·7	− 5·6		
		+ 8·2						+ 3·5	+ 1·9
20	− 2·1		+ 0·2	+ 0·5	+ 1·0	− 1·7	− 2·1		
		+ 6·7						+ 6·2	− 2·4
21	+ 4·6		+ 0·2	+ 0·5	+ 0·8	− 1·0	+ 4·1		
		+ 0·3						+ 0·3	+ 3·0
22	+ 4·9		+ 0·1	+ 0·4	+ 0·7	− 0·7	+ 4·4		
		− 2·7						− 2·9	+ 1·4
23	+ 2·2		+ 0·1	+ 0·4	+ 0·7	− 0·5	+ 1·5		
		− 1·8						− 1·8	...
24	+ 0·4		+ 0·1	+ 0·4	+ 0·6	− 0·4	− 0·3		

8. In the first column of each of Tables IV. and V. is given the divisor of the whole adopted period of 156 years, for which the Fourier coefficients in the second column are reproduced from *M.N.*, lxxiii. p. 722.

In the third column the differences are formed so that we may see which are the big terms according to the plan indicated in Table II. of the present paper. The biggest of these is −34·3 in Table IV., indicating a large term between divisors 13 and 14, which correspond to 12·00 years and 11·14 years respectively. The corresponding difference in Table V. is also large, and there can be little doubt that this is the chief term. But in Table V. the two following differences are also large. From this table alone we might assign a term between 14–15; but looking at Table IV. we see that we shall in that case require an additional term in 15–16. From economical considerations we are led to try whether such a term will not suffice, and we then see that the large value of 14–15 in Table V. may be the sum of two effects of the two neighbouring terms, both of which would act in the same direction. On trial we find that the facts can be suited in this way.

The terms 10–11 and 18–19 have large differences in both Tables IV. and V.; and there are no other large differences common to the two tables. Hence we select these four terms for trial in the first instance. The process of determining their elements by trial and error need not be followed.

In columns 4 to 7 of each table are given the solutions by trial

and error, the coefficients in each column following the law described on pp. 716–718 of the last paper; and reproduced in Table I. of the present paper.

Subtracting the sum of these four terms from the second column, we get the residues in the eighth column of each table.

In the ninth are formed the differences, and in the tenth the quantity we have called $(a-b+c)$; which is at least twice any residual coefficient. Now, inspection of these columns shows that there is no residual coefficient of a simple periodicity exceeding 3·4 in either S or C; and that the largest residuals occur in places where we have not admitted a periodicity, *i.e.* where we cannot modify an existing one, but should have to add a new one. Without claiming that the very best solution has been reached, it is clear that any improvements will modify the solution by terms which are small compared with those adopted: and as these raise questions of their own order of magnitude, we may for the present put aside possible improvements.

9. The first such question raised is, whether we are content to leave such residues as +10·6 in series C, divisor 15: or even +7·8 in series S, divisor 18. These are surely of magnitude comparable with the adopted solution?

The answer seems to be that in each case these large residuals belong to a series of consecutive terms all of the same sign, and thus of the type considered in § 2. Their combined effect will be of one sign at the beginning, near zero in the middle, and of opposite sign at the end. If the oldest observations and the most recent do not agree, we clearly have here a means of reconciling them. We shall presently see that there is such disagreement between the earliest and the latest observations: and indeed the very existence of these groups of terms is good evidence of the fact. But considering the nature of the material in the early years, the fact that there is such a difference is not surprising; and if we ascribe it to the imperfection of the early records, we may then ignore these terms of reconciliation, being content to find a consequent departure of the earlier observations from the calculated curve. It will probably be conceded that this is the proper procedure, at any rate in the first instance.

10. There is one other large term not hitherto noticed, viz. that for divisor 2, or period 78 years. The significance of this term is as follows:—A Fourier sequence consists of terms which oscillate about zero, while Wolf's numbers have zero for the minimum limit. We can make the numbers oscillate about zero by running a median line through them and adopting this as our standard of reference, so that the numbers are as often below as above it. This median line will be a long wave of period about 70 or 80 years, owing to the fact that Wolf's numbers die down about 1815, and have large peaks about 1780 and 1850; and the Fourier term above noticed represents this long wave. There is no harm in applying it if we remember its origin (which was not fully realised in *M.N.*, lxxiii. p. 122 *et seq.*), and it will

make our calculated curve fit the sun-spot curve better. But there is also no harm in leaving it out, in which case we shall merely have the calculated curve rising above the other. The latter course is preferred because of its greater simplicity. We shall therefore calculate simply the four terms of which the periods have been found above.

11. Now we can find the coefficients and phases by the formulæ given in the former paper (*M.N.*, lxxiii. p. 716). But it is simpler to go back to the observations, when once we have identified the periods. We can then include any harmonics of the main terms. For instance, we have above found a period of $156/13{\cdot}67 = 11{\cdot}41$ years, the half of which is 5·7 years; and on p. 723 of the last paper we noticed a harmonic in this neighbourhood. (In § 27 the period is actually given in error as 5·70: it should have been 5·50 years.) There is, moreover, a certain disadvantage in using the Fourier sequence for periods which do not coincide with any of its terms, viz. the phases represent a compromise. Consider, for instance, a term

$$\cos 15{\cdot}5\,\theta.$$

When $\theta = 0$ this is $+1$, and when $\theta = 2\pi$ its value is -1. But all the terms of the Fourier sequence have coincident values for $\theta = 0$ and $\theta = 2\pi$. We can only represent this term approximately and by compromise (see § 3). The phases will be right in the middle of the series, but will gradually accumulate an error of a quarter period in opposite directions at the two ends. This disadvantage can be obviated by recurring to the original observations, armed with our knowledge of the periods, and determining the amplitudes and phases afresh.

12. Another advantage may also be gained, which may be illustrated from the present example. We have periods as follows:—

Table VI.

Divisor of 156 Years.	Period in Years.	
$10\frac{2}{3}$	14·63 say $14\frac{2}{3}$ years	$7 \times 14{\cdot}63 = 102{\cdot}4$
$13\frac{2}{3}$	11·41 ,, $11\frac{2}{3}$,,	$9 \times 11{\cdot}41 = 102{\cdot}7$
$15\frac{1}{4}$	10·22 ,, $10\frac{1}{4}$,,	$10 \times 10{\cdot}22 = 102{\cdot}2$
$18\frac{2}{3}$	8·36 ,, $8\frac{1}{3}$,,	$12 \times 8{\cdot}36 = 100{\cdot}3$

Now the middle two terms "interfere" in about a century. From the relations shown in the last column it is seen that the first three terms repeat very nearly in 102·4 years. [To make all four repeat in this period we must alter the period of the last to $102{\cdot}4/12 = 8{\cdot}53$ years, or the divisor to $18\frac{1}{3}$. It can be seen on reference to Tables IV. and V. that such an alteration, while not inconsistent with the observations to the order of approximation used, would not be suggested by them: and the idea of an exact cycle of this kind is not borne out by the observations in detail.] Hence to obtain one of these terms clear of the other we ought to limit ourselves to a century of observations. We have 160 years

available. Let us therefore take the first century and the last century, and compare the results. Arranging the observations in periods of 11·4 years, we get

TABLE VII.

Means of 11·4-Year Periods.

Years.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	(12).
1749–1805	51	46	34	38	45	54	65	62	59	66	56	(62)
1806–1862	51	37	25	13	7	16	36	67	78	69	61	(63)
1863–1908	35	26	15	11	5	21	51	80	78	71	59	(50)

13. These numbers are means, to a Wolf unit, of successive *half centuries* (approximately), there being five groups of 11·4 years in each of the first two lines, and four in the last line. Either the first two or the last two combined should give us the 11·4-year periodicity clear of its neighbour the $10\frac{1}{4}$-year. Hence the first and last group should be similar; but this is not the case. The discordance can be concisely shown by a harmonic analysis of the three groups; for the purposes of illustration it will suffice to treat them as 12-year periods: and this gives

Years.	
1749–1805	$-12 \sin\theta - 5\cos\theta = 13\cos(\theta - 247°)$
1806–1862	$-31 \sin\theta + 5\cos\theta = 31\cos(\theta - 280)$
1863–1908	$-34 \sin\theta - 7\cos\theta = 34\cos(\theta - 259)$

14. The *phases* of the first and third groups are in fair accordance, which is so far good; the discordance of the middle group does not matter, as it is probably due to the "interference" of the $10\frac{1}{4}$-year period.

But the *amplitude* of the first group is conspicuously smaller. It has already been remarked by Professor Schuster (*Phil. Trans.*, 206, A, 1906) that the 11-year period died down in the early years of observation. This may possibly be due to the imperfection of the early records: the future must decide.

15. For the purposes of immediate prediction (which is one of the uses of such analysis as the present) it does not much matter whether we put the discordance down to this cause or ascribe to it a physical origin, provided the latter be due to gradual and not abrupt changes. For (*a*) if we consider the early observations imperfect, we shall determine the 11·4-year periodicity from the years 1806–1908, during which it has maintained a constant coefficient. (*b*) If we consider that the change in the coefficient is real, it has apparently increased from the first period to the second, and then become stationary. We shall therefore not make a serious error by carrying on the stationary value a little longer.

16. Hence, for the purposes of prediction we shall determine the 11·4-year wave from the years 1806-1908, and it is found to be as follows, from the mean of the nine cycles:—

44 32 21 12 6 18 43 73 78 70 60 (57),

figures which include, of course, the second, third, etc., harmonics. And we shall apply this to *all* the observations, so that the early ones will naturally appear discordant. This procedure, as remarked in § 9, is doubtless equivalent to neglecting some of the consecutive terms of similar sign.

17. Drawing a curve through these points we get corrections to the separate years, and can remove this term from the whole series. Having done so there is no longer the same reason for restricting ourselves to a century, for the remaining terms are well separated. The $10\frac{1}{4}$-year cycle comes out, in groups of 41 years—

TABLE VIII.

$10\frac{1}{4}$-*Year Period.*

Years.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	(11).
1749–1789	68	74	58	55	53	41	47	68	85	78	(34)
1790–1830	40	20	6	1	10	20	29	36	44	52	(38)
1831–1871	51	48	49	47	51	62	77	70	70	60	(47)
1872–1911	47	36	37	24	30	36	39	42	43	(41)	(25)
Mean	52	45	37	32	36	40	48	54	61	58	(36)

18. The agreement and disagreement of these groups may again be illustrated by a rough harmonic analysis, treating the period as 10 years. The results are

$$\begin{aligned}
1749\text{–}1789 \quad & -7\sin\theta + 17\cos\theta = 18\cos(\theta - 337^\circ) \\
1790\text{–}1830 \quad & -20\sin\theta + 10\cos\theta = 22\cos(\theta - 297^\circ) \\
1831\text{–}1871 \quad & -13\sin\theta - 4\cos\theta = 13\cos(\theta - 198^\circ) \\
1872\text{–}1912 \quad & -6\sin\theta + 4\cos\theta = 7\cos(\theta - 305^\circ) \\
\text{Mean} \quad & -12\sin\theta + 7\cos\theta = 14\cos(\theta - 300^\circ)
\end{aligned}$$

For the purposes of prediction there are two possible alternatives: we may treat the discordances as accidental, and apply the mean wave; or we may treat them as real, in which case the coefficient must be nearly zero at present. For further study of the other periods the simplest course is to apply to past observations the mean wave, assumed constant. Removing it on this supposition and arranging the residuals in periods of $14\frac{2}{3}$ years, there is not material for more than two good groups, from 1749–1836 and from 1837–1909, as follows:—

TABLE IX.

$14\frac{2}{3}$-*Year Period.*

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	(15).
65	50	29	29	28	31	28	32	50	58	56	54	54	62	(62)
43	37	40	41	43	37	36	36	37	58	40	49	44	35	(30)

Harmonic analysis as for 15 years gives

$$\begin{aligned}
& -17\sin\theta + 7\cos\theta = 18\cos(\theta - 293^\circ) \\
& -2\sin\theta - 2\cos\theta = 3\cos(\theta - 225^\circ)
\end{aligned}$$

19. This $14\frac{2}{3}$-year term apparently existed in the first half of the observations, but not in the second, as already remarked by Professor Schuster (*loc. cit.*). For the purposes of prediction it may be neglected: for study of the final term it was applied to the observations as a mean wave.

20. Finally, after removing all these terms as indicated, we come to the $8\frac{1}{3}$-year period. We have material for three groups of $6 \times 8\frac{1}{3} = 50$ years and 8 years over. It seems best to omit the first 8 years as probably most defective. The others give

TABLE X.

$8\frac{1}{3}$-*Year Period.*

Years.	1.	2.	3.	4.	5.	6.	7.	8.	(9).
1757–1806	19	21	26	58	83	84	71	59	(40)
1807–1856	26	22	23	36	45	47	41	39	(36)
1857–1906	36	40	42	38	39	27	31	32	(39)
Mean	27	28	30	44	56	53	48	43	(38)
Mean of all	29	31	33	44	53	48	42	37	(38)

In the line "mean of all" the first 8 years have been included, and it will be seen that they are discordant enough to alter the mean appreciably.

21. Forming the first harmonic as before, we get

Years.	
1757–1806	$-23 \sin\phi - 29 \cos\phi = 36 \cos(\phi - 220°)$
1807–1856	$-10 \sin\phi - 9 \cos\phi = 13 \cos(\phi - 235)$
1857–1906	$+6 \sin\phi + 0 \cos\phi = 6 \cos(\phi - 90)$
Mean	$-9 \sin\phi - 13 \cos\phi = 16 \cos(\phi - 215)$

22. The third group is discordant in both phase and amplitude: but we can force it into line by altering the phase 180° and changing the sign of the coefficient, assuming that the coefficient is a linear function of the time. The successive coefficients are then $+36$, $+13$, and -6, which are sensibly in arithmetical progression. The form $a - bt$ must then be regarded as the best approximation we can make, from available material, to a term of long period: which is equivalent to regarding the periodicity as a pair of terms, both of which fall between two Fourier terms. Until we have either independent evidence of such a pair of terms, or continued evidence of their co-existence, we can scarcely regard them as real. But before leaving them we may note the way in which they manifest themselves in the Fourier sequence. Looking at the second columns of Tables IV. and V. we see that the S series has the larger term at divisor 18, and the C series at divisor 19; *i.e.* though both S and C agree in putting a term in between 18 and 19, S puts it nearer 18 and C nearer 19. The pair is thus of the type

$$\begin{aligned} &\sin(q + x - y)\theta + \cos(q + x + y)\theta \\ &= \cos\{(q + x - y)\theta - 90°\} + \cos(q + x + y)\theta \\ &= 2\cos(y\theta - 45°) \cdot \cos\{(q + x)\theta - 45°\}. \end{aligned}$$

Here x and y are fractions, and y is less than $\frac{1}{2}$. Hence $y\theta - 45^\circ$ ranges within experience from -45° to $<135^\circ$: so that $\cos(y\theta - 45^\circ)$ is positive for most of the time, ending up as a small negative.

23. We have now reviewed all the terms, and in no case, not even that of the well-known 11-year period, is there satisfactory evidence of a periodicity remaining stable throughout experience; though the 11-year period has been satisfactorily constant for the last century, and the imperfection of the early records may conceivably be the reason of its apparent failure from 1750–1800.

The companion $10\frac{1}{4}$-year period has at present almost died out, though it was conspicuous from 1749 to 1830, and perhaps later.

The $14\frac{2}{3}$-year period, strong in the first 75 years, gives no sensible coefficient for the last 75 years. On scrutinising the individual cycles there seem to be signs of a revival, but they are not as yet very strong.

The $8\frac{1}{3}$-year period has steadily run down hill, and has now changed sign to a small negative coefficient.

This result has been a disappointment. Had even one of these smaller terms shown signs of a sensible coefficient at present, we might have made two alternative predictions, one treating it as spurious and the other as real: and the event might have thrown some new light on the problem. As it is, prediction for the immediate future would on either supposition be practically confined to the 11·4-year period.

[*Note added November* 24.—The above represents the results of attempts to treat the periodicities as permanent, and the small profit that comes from regarding them as slowly varying. At the time when it was written the plan of trying whether discontinuities could be traced had already been formulated, but not much had come of it. On further trial, however, a new method suggested itself which not only makes manifest the discontinuities but suggests a *vera causa* for them. This must, however, be reserved for a future paper.]

Sixth Note on the Number of Faint Stars with Large Proper Motions. Zone $+29^\circ$.

By F. A. Bellamy, M.A.

This brief note is the outcome of a comparison of two plates exposed with the Oxford Astrographic Telescope of 13-inch aperture. One plate (542) was exposed on 1894 Oct. 2, and the measures have been printed in vol. iii. p. 169; the other plate (2884) was exposed on 1911 Dec. 4 (interval 17·17 years), and it has been measured in the direct and reversed positions by Mr. R. J. Pocock, B.A. (Queen's College, Oxford), on 1912 July. The computations and the comparison of the two plates were also performed by him. The examination of certain discordances and the deduction of the results given in this note were made by myself.

The comparison has been carried out on the same lines as with the five previous notes (*M.N.*, vols. lxix., lxxi., and lxxii.).

The number is from the Oxford Astrographic Catalogue, vol. iii., centre Dec. + 29°, R.A. $22^h\ 7^m$; the R.A. and Dec., also the magnitude, are deduced for each star from that plate (542). Δx and Δy are the differences between the two plates expressed in réseau intervals (unit ·001 = 0″·3); and the centennial P.Ms. are also given.

No. in +29°.	Photo. Mag.	R.A. (1900·0).	Dec. (1900·0).	Δx.	Δy.	Centennial P.M. in Arc.
		h m s	° ′ ″			″
64504	11·6	22 4 1·75	+29 13 59·8	− 2	−10	17·8
64566	10·7	22 4 51·03	29 36 29·6	− 5	− 2	9·4
64552	10·9	22 7 0·07	29 34 2·9	− 2	− 5	9·4
64388	8·7	22 7 31·31	28 34 8·8	+11	+ 7	22·8
64336	10·1	22 9 35·75	28 13 6·8	+ 5	+ 5	12·4

No. 64566 is Cambridge A.G.C. 13230. It is a single observation in that catalogue, but has been reobserved since. No. 64388 is Cambridge A.G.C. 13269.

There were 337 stars measured on the first plate, which gives 4·4 times B.D.'s number; and 575 stars on the second plate.

University Observatory, Oxford:
1913 *August* 15.

Seventh Note on the Number of Faint Stars with Large Proper Motions. Some A.G.C. Stars in Zone +25°. By R. J. Pocock, B.A., B.Sc., Queen's Coll., Oxford.

(*Communicated by Prof. H. H. Turner.*)

The following proper motions were determined in the course of the reduction of a plate (No. 2901 in the Oxford series), exposed on 1912 Oct. 24, with 31 exposures of overlapping regions differing in R.A. by 2 or 3 minutes, and with plate centres ranging approximately from $21^h\ 4^m$ to $22^h\ 28^m$ in zone +25°.

The methods of reduction were, in the main, similar to those described by Professor Turner in *Monthly Notices*, 1911 Dec.

The Proper Motions are determined from a comparison of the places determined from Plate 2901 with the mean places found from previous Oxford plates on which the stars occur. In this respect the methods used in this work differ from those used in the work of which results have been given in previous papers of this series.

In addition to determining Proper Motions, the places of all

the stars on Plate 2901, which are in either the Cambridge or Berlin A.G. Catalogues, were used for re-solving some of the old Oxford astrographic plates. There were accidental reasons preventing the use of fainter stars. A short summary of the results obtained is included.

The Proper Motions are exhibited in Table I. The Oxford numbers are taken from vol. vii. of the Oxford Astrographic Catalogue. Some of the stars occur also on plates in zone $+26°$, but the corresponding Oxford numbers are not given. The photographic magnitude is determined from the formula $a - b\sqrt{d}$, using the constants a, b, printed in the Astrographic Catalogue for each plate.

It is to be particularly noted that the unit in columns 7 and 8, as throughout the paper, is ·0001 of a réseau interval ($=0.''03$), not ·001, as in the previous papers of this series.

TABLE I.

List of Proper Motions.

Ref. No.	A.G.C. No.	Oxford. No. +25°.	Oxford. Photo. Mag.	No. of Plates.	Mean Interval in Years.	Δx. (Unit = ·0001.)	Δy. (Unit = ·0001.)	Cent. P.M. in arc.
								″
1	B 8525	76440	3·2	2	15·1	+217	+ 21	43·3
2	C 13020	75309	8·5	7	14·6	+158	+ 14	32·4
		76105	8·1					
3	B 8432	75047	8·8	2	14·5	−119	− 83	30·0
4	C 12550	73560	8·3	4	17·6	− 87	−129	26·5
5	C 12808	74873	10·6	3	15·4	−131	+ 6	25·5
6	B 8616	78044	8·9	2	11·1	− 70	− 40	21·8
7	B 8578	77115	8·8	2	11·1	− 77	− 9	20·9
8	B 8196	73174	9·2	1	19·1	−128	+ 35	20·8
9	C 12875	74660	6·5	4	13·3	+ 76	+ 29	18·4
10	B 8201	73175	8·8	1	19·1	− 80	− 77	17·5
11	C 13283	76632	6·9	5	13·9	− 14	− 76	16·7
		77538	6·7					
12	B 8213	73200	8·2	2	19·1	+ 89	− 1	14·0
		73514	8·0					
13	C 12834	74970	10·0	6	14·1	+ 30	+ 58	13·9
14	C 13492	78979	8·2	3	15·7	+ 70	+ 20	13·9
15	B 8247	73433	9·4	1	19·1	− 7	− 85	13·4
16	C 13123	76044	9·4	2	16·6	+ 60	+ 37	12·8
17	C 13340	77722	7·8	4	14·5	+ 32	+ 51	12·5
18	C 12352	73350	8·0	3	18·1	+ 72	+ 15	12·2
19	C 13472	78224	9·0	5	13·8	+ 53	+ 17	12·1
		78972	8·8					
20	C 12656	74162	8·4	2	18·1	+ 62	+ 30	11·4

TABLE I.—*List of Proper Motions*—Continued.

Ref. No.	A.G.C. No.	Oxford. No. +25°.	Oxford. Photo. Mag.	No. of Plates.	Mean Interval in Years.	Δx. (Unit = ·0001.)	Δy. (Unit = ·0001.)	Cent. P.M. in arc.
21	B 8192	73126	9·1	1	19·1	+ 14	− 70	11″·2
22	B 8585	76937	8·5	2	11·1	− 32	+ 26	11·2
23	C 12679	74176	10·0	2	18·1	+ 56	+ 30	10·5
24	C 12298	72761	7·3	4	16·9	− 49	− 29	10·1
		73242	10·0					
25	C 12873	74769	8·7	4	13·3	− 40	+ 18	10·0
26	C 13113	76097	7·8	2	16·6	+ 27	+ 44	9·3
27	B 8165	73191	8·3	1	19·1	+ 58	− 10	9·2
28	C 12466	73325	9·4	4	18·1	− 47	+ 17	8·3
		73651	9·3					
29	C 13098	75827	9·2	3	18·4	− 33	− 29	7·2
30	C 12865	74919	9·9	4	13·3	+ 30	+ 8	7·0
31	C 12757	74171	8·9	2	18·1	− 1	− 32	5·3
32	B 8451	75143	9·3	2	14·5	+ 20	− 15	5·2
33	C 12592	73673	8·6	2	18·1	+ 31	+ 3	5·1

The smaller Proper Motions would not have been included were they not confirmed by a comparison with the Cambridge and Berlin Catalogues.

Nos. 4, 9, 24, above, are B 8245, B 8389, B 8145 respectively.

Notes on Nos. 2, 9, 11, 13, 14 will be found in vol. viii. part 1 of the Oxford Astrographic Catalogue (now in the press). No. 18 occurs in List B in vol. viii. as a Proper Motion, requiring confirmation, which is now supplied.

Proper Motions of Nos. 1, 3, 7, 8 are given in the Berlin A.G.C. from comparisons with earlier catalogues.

No. 9 occurs in Boss's P.G.C., centennial Proper Motion 16″·3. Proper Motions of C 12462, C 12847 (κ Pegasi), C 12867 are given in Boss, but they are too small to be determined in the present work. The following stars in this region are suspected of Proper Motion, but further confirmation is needed.

C 12301, C 12364, C 13008, C 13022, C 13271, C 13560, C 13561, C 12750, B 8185, B 8295, B 8380, B 8406, B 8440.

Eleven plates with centres in zone +25°, R.A. $21^h\ 8^m$ to $22^h\ 28^m$, were re-solved, the Cambridge and Berlin A.G.C. places being corrected for Proper Motion to 1900·0, as determined from Plate 2901. Further, the Cambridge stars were corrected for magnitude equation in R.A., as determined at Oxford (Astrographic Catalogue, vol. viii.); the Berlin stars were corrected for the systematic difference between the two catalogues, as determined from Plate 2901. This difference was found to be Δ R.A. = +19, Δ Declina-

tion = − 3, the unit being 0·0001 of a réseau interval (= 0·″03). Auwers' determination (*Ast. Nach.* 3842) gives

R.A.	Δ R.A.	Δ Decl.
20^h	+16	+ 5
21^h	+22	− 10
22^h	+26	− 4

in the same units. Finally, residual errors in the A.G.C. places, as determined from the present work, were also applied. The values of A − E, B + D, so obtained, are given for comparison with the same values for the old solutions in Table II. It will be noticed that the values of A − E are very considerably improved. The values of B + D are not a great deal better numerically, and still tend to be systematically negative, though not to the same extent as before.

TABLE II.

Comparison of Old and New Values of A − E, B + D.

Plate No.	Old Solution.		New Solution.	
	A − E.	B + D.	A − E.	B + D.
465	− ·00051	− ·00005	− ·00023	− ·00004
466	+ ·00001	− ·00013	+ ·00019	− ·00010
460	− ·00021	− ·00020	− ·00011	+ ·00008
471	+ ·00005	− ·00045	+ ·00013	− ·00011
2290	+ ·00015	− ·00034	+ ·00014	− ·00015
2291	+ ·00004	− ·00015	− ·00007	− ·00034
1210	+ ·00014	− ·00019	+ ·00008	− ·00031
2384	− ·00011	− ·00032	+ ·00001	− ·00021
2385	+ ·00010	− ·00007	− ·00008	− ·00015
2386	+ ·00021	− ·00003	− ·00010	+ ·00016
2387	− ·00045	− ·00012	− ·00002	+ ·00016

The large values of B + D for Plates 2291, 1210 may be partially accounted for by the fact that two stars common to these plates are probably affected by Proper Motion, though there was not sufficient evidence to justify its application. As the number of stars was already small, these two were retained, though, with more stars available, they might have been omitted.

The new plate constants represent an undoubted improvement over the old, owing to the application of corrections, as already mentioned. They are probably capable of further improvement by

(1) Applying the magnitude equation (which is known to be small) to the Berlin stars and to the Cambridge Declinations.

(2) Use of more and fainter stars.

The residuals obtained with these constants are a considerable improvement on those previously obtained. They are collected

(without regard to sign) in Table III. The numbers in the 2nd and 3rd lines give the total number of residuals in x and y respectively not greater numerically than the corresponding number in the 1st line.

TABLE III.

Residuals obtained with the New Plate Constants (*unit* ·0001).

Residual >	5	10	15	20	25	30	35	40	45	50	Total.
No. in x	54	96	135	168	200	212	236	246	254	258	266
No. in y	64	111	152	193	220	241	249	257	262	264	266

The majority of the larger residuals in Table III. are either stars suspected of Proper Motion, but without sufficient evidence to justify its application, or are on the older plates, which were measured in only one position and by several different measurers. Re-measures of one of these plates showed that several of the larger residuals were due to this cause. It should be remembered that both on Plate 2901 and on the old plates the unit of measurement is equivalent to 10 in the above table. Thus, two-fifths of the stars in each co-ordinate give a negligible residual.

Moreover, since R – D discordances of ± 40 (in the above units) are usually left unrevised, only about one-fifth give residuals, which cannot be entirely due to the measures. Comparing with the residuals from the old solution, there is practically no change in y, but the residuals in x are greatly improved. The actual number of stars used in the solutions was not quite the same, but a factor was applied to the numbers of residuals from the old solution to make them comparable with the numbers in Table III. Table IV. gives a comparison of the two sets of residuals. Lines 2 and 3 give the number of residuals in the old and new solution, which exceed the corresponding number in the 1st line.

TABLE IV.

Comparison of Residuals in x from New and Old Solutions (*unit* ·0001).

Residual >	10	20	30	40	50
New Solution	170	98	54	20	8
Old Solution	187	142	108	76	55

The distinct improvement is obvious. Moreover, this improvement is not due to the application of Proper Motions, but to the elimination of errors in the meridian catalogues, as is clear from the following considerations:—

(1) There is no special improvement in the y residuals as there should be if the difference were due to Proper Motions alone.

(2) Only 33 stars out of a total of 229 used in the re-solutions show sensible Proper Motions.

University Observatory, Oxford:
1913 *November* 4.

On the Occurrence of Europium Lines in Stellar Spectra.

By F. E. Baxandall, A.R.C.Sc.

(*Communicated by Professor Newall.*)

In a letter to *The Observatory*, dated October 18, it was pointed out that several lines which Professor Belopolsky * has shown to have a periodic variation in intensity in the spectrum of α Canum Venaticorum, agreed very closely in wave-length with the strongest lines of europium, and that this suggested a real identity between the two sets of lines. The facts may be concisely re-stated here for convenience of reference. The following table gives a comparison of the wave-lengths of lines in Belopolsky's list for α Canum Venaticorum, and those of the strongest lines of europium as given by Exner and Haschek in the same region of spectrum.

Strongest Europium Lines (Exner and Haschek.)			α Canum Venaticorum Lines (Belopolsky).	Diff. in λ.	Remarks.
	Intensity.				
λ.	Spark.	Arc.	λ.		
3930·65	50	50	3930·67	0·02	...
3972·16	50	50	...	...	H_ϵ would mask this line in star.
4129·90	100	100	4130·04	0·14	..
4205·20	50	100	4205·20	0·00	...
4435·75	30	50	4435·77	0·02	Stellar line partly due to Ca 4435·85.
4522·76	15	20	4522·87	0·11	Stellar line probably chiefly due to *p* Fe line 4522·80.
4594·22	10	100	*4594·26	0·04	...

* This line was not given in Belopolsky's published list, but occurs in a further list of lines sent to the Solar Physics Observatory.

It will be seen that the mean difference between the stellar wave-lengths and those of the apparently corresponding metallic lines is only 0·055 tenth-metres. In the above list lines 4130·04 and 4205·20 cannot be adequately accounted for by lines of elements other than europium. These are the two lines which show the most conspicuous changes of intensity in Belopolsky's spectra.

The presence of europium lines in celestial spectra has previously been referred to by Professor Dyson, who, in his publication on eclipse spectra,† made several identifications of europium lines with chromospheric lines, some of which could

* *Bulletin de l'Académie Impériale des Sciences de St. Pétersbourg*, 1913, p. 689 *et seq.*

† *Phil. Trans.*, vol. ccvi. pp. 403–452.

very well be attributed to lines of other metals, but others had no chemical origin suggested for them other than europium. In Jewell's published results on eclipse spectra,* identifications with europium lines were also suggested.

In a paper "On the Presence of Europium in Stars," † Lunt gave reasons (based on discordant radial-velocity measures on a certain line in the spectrum of Arcturus which was assumed to be identical with the solar line 4435·851, attributed to calcium) for suspecting that a contaminating line existed on the violet side which, blended with the calcium line, was the cause of the discordant results. He found that there is a strong europium line at such a wave-length as would account for the result he obtained, assuming it to be equal in intensity to the calcium line, and existing alongside it in the stellar spectrum, and concluded that europium was really represented in the stellar spectrum. This conclusion was arrived at before Dyson's publication appeared, but Lunt's results were published subsequently to those of Dyson.

With these results in view, and the increased importance which europium lines attain in stellar spectra if Belopolsky's special lines are really due to that element, it seemed desirable to investigate whether they occur in stellar spectra of other types, and to review the evidence for europium in celestial spectra in general. This has been undertaken so far as the available material at Cambridge will allow. In an inquiry of this nature, especially as the wave-lengths of some of the strong europium lines agree so closely with those of lines in stellar spectra which belong chiefly to other metals, it is necessary to use spectra showing good dispersion and definition. The spectra investigated, therefore, have been those in the Kensington series which were obtained with two six-inch Henry prisms and six-inch objective, and the Cambridge four-prism spectra used for radial velocity work. The former give a dispersion at Hγ of 9·5 Angström units per millimetre, and the latter 6·5 units.

No evidence was found for europium lines in spectra of an earlier type than α Canis Majoris (A_0). The most suitable spectra for investigation in the A and F types are α Canis Majoris (A_0), α Cygni A_2(Pec), and α Canis Minoris (F_5). Out of the many lines in the europium spectrum, only those lines which occur in the region available for a study of stellar spectra, and show an intensity of 10 or higher in arc and spark (Exner and Haschek), have been considered in this discussion.

In the following tables, under the heading "Stellar Line," M signifies that the wave-length has been obtained from micrometer measures on the line and on stellar reference lines of known origin; E that the wave-length has been estimated by eye from closely adjacent stellar lines; and K that the wave-length is extracted from previously published stellar records (Kensington).

* *Publications of the United States Naval Observatory*, 2nd series, vol. iv., Appendix I. p. D129 *et seq.*

† *Roy. Soc. Proc.*, vol. lxxix. p. 118.

α Canis Majoris (A_0).

Europium (Exner and Haschek).	Stellar Line.	Reference Lines (Stellar).	Remarks.
3930·65	3930·52 M	*p* Ti 3913·61 Al 3961·67 *p* Ti 4012·54	Probably a blend of Eu and Fe 3930·45.
3972·16	...	...	Masked by Hε.
4129·90	4129·91 E	Si 4128·20 Si 4131·04	Faint, probably Eu only.
4205·20	4205·21 E	Fe 4199·27 Fe 4202·20	Faint, probably Eu only.
4435·70	...	...	Doubtful in stellar spectrum.
4522·70	4522·7 K	...	Chiefly *p* Fe 4522·80, but possibly a blend with Eu.
4594·22	4594·18 E	*p* Cr 4588·38 *p* Cr 4592·25	Solar lines (Rowland) at 4594·113(0) unassigned ·207(2) V. Stellar line probably Eu.

In summing up the evidence for and against the occurrence of europium lines in the spectrum of α Canis Majoris, it may safely be concluded that they do exist in the stellar spectrum, but only as weak lines. Certainly several of the stellar lines which fall near the position of the europium lines can be ascribed to other elements known to be represented in the star, but there are other lines, such as 4129·9, 4205·2, 4594·2, for which identifications with lines of other elements cannot be claimed. A further study of the Sirius spectrum with equal dispersion to that employed by Belopolsky on his spectrum of α Canum Venaticorum—or even greater—would probably throw further light on the question under discussion.

α Cygni (A_2 Pec).

Europium (Exner and Haschek).	Stellar Line.	Reference Lines (Stellar).	Remarks.
3930·65	3930·52 M	*p* Ti 3913·61 *p* Ca 3965·63 *p* Ti 4012·54	Possibly blend of Eu and Fe 3930·45.
3972·16	...	...	No evidence for stellar line.
4129·90	...	...	No evidence for stellar line.
4205·20	4205·2 K	...	Possibly blend of Eu and *p* V 4205·24.
4435·70	...	...	No evidence for stellar line.
4522·70	4522·7 K	...	Strong stellar line, chiefly *p* Fe 4522·80. Possibly blend of Eu and *p* Fe.
4594·22	4594·42 E	*p* Cr 4592·25 Un 4596·6	Possibly Eu.

From the preceding table it is clear that the evidence for the presence of europium lines in the α Cygni spectrum is unconvincing. The absence of the strong lines 3972·16, 4129·90, 4435·70 suggests that europium is not represented. Other α Cygni lines occurring at or near the europium wave-lengths can be fairly well accounted for by lines of other elements known to be represented in the spectrum.

α *Canis Minoris* (F_5).

Europium (Exner and Haschek).	Stellar Line.	Reference Lines (Stellar).	Remarks.
3930·65	...	...	Masked by K line of calcium.
3972·16	...	...	Masked by H line of calcium and H_ϵ.
4129·90	4129·6 K	...	Weak stellar line apparently more refrangible than Eu line.
4205·20	4205·10 M	Fe 4199·27 Fe 4202·20	Possibly Eu only.
4435·70	4435·9 K	...	Probably chiefly Ca 4435·85. Possibly Eu also.
4522·70	4522·9 K	...	Probably chiefly *p* Fe 4522·80 Possibly Eu also.
4594·22	4594·3 K	...	Possibly Eu only. Evidence for V 4594·27 weak.

Summing up the evidence for the occurrence of europium lines in the spectrum of Procyon, it may be said that although the europium lines agree closely in wave-length with stellar lines, the latter can be accounted for to a great extent by lines belonging to metals known from other evidence to be present in the absorbing vapours of the star, and thus the evidence for europium is at best only presumptive, and in no way conclusive. The absence of a line agreeing in position with the strong europium line 4129·90 rather raises a doubt whether europium is represented in the stellar spectrum at all. It would be of interest to know, however, from anyone who has the available material to test the point thoroughly, whether radial-velocity measures on the stellar-Ca line 4435·85, when treated as a "pure" Ca line, give discordant results as compared with measures on other uncontaminated lines, in the way which Lunt found for Arcturus.

The results for α Canis Minoris are rather surprising, as the metallic line absorption of the star has been shown to resemble closely that of the chromosphere, and if the europium lines really occur in the latter spectrum, as recorded by Dyson, one would expect to find them in that of the star.

The three stars already investigated are, in the Harvard classification, of the A_0, A_2(Pec), and F_5 types respectively. As Lunt has given the evidence for a star (Arcturus) of the K type, it remains now to consider the best spectra available of the intermediate types. For the G type (true solar) Lunt has recorded

that the results he obtained from measures on the Ca line 4435·85 were normal and not discordant, so he presumably suggests that europium lines do not exist, or are not strong enough for measurement, in the spectra of true solar stars. Although, owing to the very great number of faint lines in Rowland's solar tables, a comparison of stellar or laboratory lines with the solar lines may easily lead to false conclusions as to identity, it has been thought worth while to make such a comparison for europium lines, to see if any good evidence is available for their existence in the solar spectrum. The comparison is given in the following table :—

Comparison of Strongest Lines of Europium with Rowland's Solar Lines.

Europium Lines.			Nearest Solar Line (Rowland).		Rowland's Origin.	Remarks.
λ. Exner and Haschek.	Intensity. Spark.	Arc.	λ.	Int.		
3688·59	10	20	3688·558	4	Ni	
3725·08	20	30	3725·090	0	...	
3819·81	50	50	3819·829	1	C	
3907·30	30	30	3907·251	0	...	
3930·65	50	50	3930·654	0	...	
3972·16	50	50	3972·134	0	...	
4129·90	100	100	4129·882	1	...	
4205·20	50	100	4205·186	1	...	
			4205·239	1	...	This solar line probably due to protovanadium.
4435·75	30	50	4435·851	4	Ca	Solar line undoubtedly chiefly due to Ca.
4522·76	15	20	4522·691	0	? Fe	
			4522·802	3	...	This solar line chiefly due to *p* Fe 4522·80.
4594·22	10	100	4594·113	0	...	
			4594·297	2	V	
4627·39	8	100	4627·392	00	...	

The preceding comparison would appear to show that there is fairly strong evidence that europium is one of the constituent metals in the absorbing atmosphere of the sun. Seven out of twelve lines agree closely in wave-length with rather weak solar lines which have not been identified by Rowland. The others also agree closely in position with solar lines to which Rowland has attached an identification, and may thus constitute blends in the solar spectrum with these other lines of Ca, Ni, V, etc.

It may be said here that if Lunt's conclusions as to the existence of europium lines of moderate intensity in Arcturus are correct, and also the present conclusions as to their occurrence in α Canum Venaticorum and α Canis Majoris, it is somewhat surprising to find the evidence for them in intermediate types (such

as α Cygni and α Canis Minoris) so weak. In going through the previously established sequence of spectra in various classifications, it is generally found that lines belonging to a particular element first appear weak, then strengthen to a maximum, and then decline in intensity to extinction.

The general results of this investigation are best given in a short table.

Star.	Harvard Type.	Evidence for Europium Lines.
Stars of type earlier than α Canis Majoris (A_0)	...	None.
α Canum Venaticorum	A (Pec)	Strong.
α Canis Majoris	A_0	Fairly strong.
α Cygni	A_2 (Pec)	Weak and inconclusive.
α Canis Minoris	F_5	Weak and inconclusive.

There is also fairly strong evidence for the occurrence of europium lines in the Fraunhofer spectrum, but as the results for the sun are based on a comparison of the metal lines with Rowland's solar lines, the order of evidence is scarcely comparable with that obtained for the stars, so this result has not been included in the above table.

Solar Physics Observatory, Cambridge:
1913 *November* 10.

Reply to Mr. Denning's "Observations of the Orionids."

By Charles P. Olivier.

(*Communicated by Cleveland Abbe.*)

In the *Monthly Notices of the Royal Astronomical Society* of June, 1913, pp. 667–672, there is an article by Mr. W. F. Denning entitled "Observations of the Orionids." This article is an extended criticism of a paper published by me in *Transactions of the American Philosophical Society*, N.S., vol. xxii. part i., entitled "175 Parabolic Orbits and other Results deduced from over 6200 Meteors." The number of points taken up by Mr. Denning being rather large, I shall reply to them approximately in order.

The first thing to be pointed out is that the word "error" as used by Mr. Denning simply means that my results differ from his own. The more my conclusions differ, the greater the "errors." The reasons why some of my results radically contradict his own are due chiefly, I believe, to the fact that my methods of reduction were the more rigorous of the two.

For instance, my observations of successive nights were never combined; the dates were given in G.M.T. to hundredths of a day; the orbits and radiants were given in each case along with the longitude of the meteoric apex, L, which furnishes the only accurate way of comparing the results of different years. Further, the actual orbits were calculated in 175 cases, etc.

To claim that my positions are correct in absolutely every case would be absurd, but I am certain that to combine my radiants into one as he proposes (p. 667, l. 34 and p. 668, ll. 32–40) would lead to error, as a single glance at the original maps and records would show.

Mr. Denning states that radiants Nos. 109 to 115 inclusive were observed by me. Actually of these 7 radiants only No. 110 and No. 114 were based on my observations, the other five having been found by Mr. J. B. Smith and Mr. J. P. Smith. Inadvertently, on p. 26, l. 3 the initials [J.B.S.] were omitted, but in the discussion on p. 15 the statement is correctly made that No. 113 was observed by Mr. J. B. Smith. One has only to look on p. 17, first table, to see that Nos. 112, 113, and 114 were considered to be the same radiant, but, being determined by three different persons, each position was there given. Had Mr. Denning, therefore, read the article more carefully he could not have made either of his statements.

I cannot admit that errors of observation were sufficient to make it necessary to combine No. 110 and No. 114, as they were 5° apart. As to the statement on p. 668, ll. 28–32, the "significance' of the "excellent agreement" of these radiants with Mr. Denning's is nothing strange, since all there mentioned were observed on Oct. 18–20 inclusive, while the maximum falls on Oct. 19. And surely it was never claimed that Mr. Denning could not properly determine a radiant from a large number of meteors, when he was willing to limit himself to the observations of a single night.

It would hence be strange if we did differ on positions secured just at the maximum.

As for his further statement on p. 668 that all my positions for 10° around his (assumed) radiant should be combined into one mean, I consider it impossible for anyone who has observed meteors to any extent to make such errors If my work, or that of anyone else, was so inaccurate that radiants 20° apart must be combined to get a mean position, then the work would not be worth discussion.

In this connection, if the Orionids, as he states on p. 667, have no very special characteristics and there are contemporary showers in the same region of the sky, how is Mr. Denning sure that some of the other radiants do not really belong to the Orionids?

Finally, as Mr. Denning has never seen a meteor map of mine, it is rather difficult for him to judge of the accuracy of my plotting, on which the question mainly depends. Also he has totally failed to answer my former criticisms of his methods on combinations for the Orionids.

His remarks on p. 670 about my Perseid observations are entirely too general to permit any definite reply. I will add that it would have given me pleasure to use his latest ephemeris for purposes of comparison with my positions had my attention been called to it. How this can affect my results, however, I fail to see. I did not include the orbits he mentions on p. 670 among

the Perseids, because most of them could scarcely be identical, as the elements show.

Mr. Denning's statement on p. 671, ll. 5–7, is rather unfair. Out of 175 orbits 77 were based on 5 or more meteors. The following table explains the rest:—

Determined by	Number of Orbits.	Confirmed by C.P.O.	Confirmed by others.
– meteors	4	0	0
2 ,,	5	4	0
3 ,,	35	7	13
4 ,,	54	17	10
Total:	98	28	23

It will hence be seen that out of 98, 28 had other positions confirmed by me, and 23 were confirmed by others. Therefore, at most, only 47 radiants remained then unconfirmed, and Mr. Denning's statement falls from "over half" to 47 out of 175. Furthermore, out of the 53 other radiants on p. 12 and p. 21, of which he makes no mention, 41 were determined by 5 or more meteors. Of the remaining 12 no less than 3 were given by stationary meteors. In further reply to his warnings, etc., about accepting positions secured by 3 or 4 meteors on one night, I cannot do better than refer to a recent article in *The Observatory*, No. 464, pp. 334–340, by Mr. Denning himself. Consult p. 338, first part of table in his article.

Nineteen radiants in all are given, only 3 determined on one night. The average for the other 16 is 6 nights interval and 5·25 meteors per radiant. Incidentally no less than 7 are determined by only 4 meteors!

I will be satisfied with the verdict of astronomers in general, after they have considered the comparative value of a set of radiants deduced from only 3 or 4 meteors on *one night*, and of a set determined by 4 to 8 meteors scattered over 2 to 10 nights. However, only 1 radiant of Mr. Denning's set was determined by 8 meteors, only 4 by 6 or 7 meteors. As to not combining more than one night's work provided the shower be a rich one (p. 671, l. 17), why, if Mr. Denning agrees with me, has he not carried this into practice? See his *General Catalogue*, pp. 246–7. Out of 57 positions for the Orionids there given, 25 were *not* determined by observations of only one night of one year. Of these Nos. 6, 9, 23, 32, and 36 were from his own observations.

Mr. Denning quotes me as saying "there is no motion in the radiant of the Leonids" (p. 672, ll. 14–15). My exact words as published were as follows (p. 20, ll. 10–12):—"Referring to the table of radiants, it will be seen that no certain movement from date to date is indicated. But since L changes only a little over 3° for the extreme dates, this is hardly to be wondered at." Hence what he adds about the Leonids has not the least bearing upon any statement of mine.

In reply to the remarks on p. 671, ll. 29–47, it can be stated

that my observations were made on 51 different dates, distributed through 7 months. This does not refer to the number of nights on which observations were made, which would be vastly greater, because meteors were frequently observed on the same date for many successive years. Does it not seem strange that if stationary radiants are fairly numerous, as Mr. Denning claims, so little indication of their existence should have been found with all this opportunity?

It may be added that some 300 meteors observed in October 1911 and 1912, as yet unpublished, confirm my former conclusions.

Yerkes Observatory,
Williams Bay, Wis.:
1913 *Aug.* 21.

Photographic Magnitudes of 262 Stars within 25′ of the North Pole. By S. Chapman, B.A., D.Sc., and P. J. Melotte. (Plate 1.)

(*Communicated by the Astronomer Royal.*)

1. The present work arose out of the need for the determination of the photographic magnitudes of a number of faint stars in certain special regions, for the standardisation of the star counts made on the Franklin-Adams' charts in the possession of the Royal Observatory. The standardisation of about thirty of these plates has been performed with the aid of the 30-inch reflector, by successive 30-minute exposures on pole and field when at equal altitude; but on account of (1) difficulties in maintaining the focus of the reflector, (2) the smallness of its field of good definition, and (3) the scarcity of nights sufficiently good for such long comparison exposures, the use of the reflector for the purpose has been abandoned. The 26-inch refractor is now used, and as regards focus and field of good definition (up to 25′ from the centre, at least) is entirely satisfactory; its grasp and transmission of light to the photographic plate is somewhat inferior to that of the reflector, but this is compensated for by the improvement in the images. The scarcity of nights suitable for long comparison exposures remains, however; and to obviate the necessity for such comparisons, another method has been adopted, which depends on the use of a wire diffraction grating.

2. The present procedure is to make a comparison of field and pole by four exposures on pole, field, field and pole (in this order) of six minutes' duration in each case.* The duplication of exposures checks the uniformity of the night and helps to detect

* It may be remarked here that a careful investigation has shown that there is no systematic difference in the images of the stars of the polar region, as obtained by the first and last exposures. Such a systematic difference has been reported elsewhere, but if any difference occurs on the Greenwich plates, its magnitude is certainly less than $0^m{\cdot}01$.

spurious images. With the 26-inch refractor, an exposure for six minutes (Ilford Monarch plates being used) gives measurable images of 13·5 magnitude stars (on the scale of Professor Pickering's north polar sequence—Harvard Circular, No. 170), in general, ranging from $13^{m}\cdot0$ to $14^{m}\cdot0$ on specially poor or good nights. The magnitudes of the field stars down to this limit of faintness are thus determined by direct comparison with the pole. Another series of plates is taken with a 60-minute exposure on the field alone, with a diffraction grating placed in front of the object glass of the telescope; this grating produces diffraction images on each side of the images of the brighter stars, with a definite magnitude interval between the central and diffraction images. A knowledge of this interval and of the magnitudes of the brighter stars (down to $13^{m}\cdot5$, say, as determined from the short-exposure plates) makes it possible to determine the magnitudes of the fainter stars on the plate, without further comparison with the pole. The diffraction grating is also used in taking the short-exposure plates, partly to determine its magnitude interval, and partly to measure the bright stars on the field (*see* the next paper for a description of the two gratings used, and a discussion of some results obtained with them). Both of the above sets of plates are being taken in duplicate. The fields to be dealt with include about 100 Franklin-Adams' regions, together with all Kapteyn's selected areas in the northern hemisphere. Short-exposure plates only are being taken for the Kapteyn areas.

3. The measures are made by simple comparison with a scale of numbered and graded comparison stars, the smallest of which is quite faint and grey. The method of measuring the diameters of the images seems to be unsuitable for the determination of the magnitude of faint stars, since to obtain images sufficiently black to have a measurable edge requires a longer exposure than can be afforded for the single-exposure plates (going down to $15^{m}\cdot0$ or $15^{m}\cdot5$), or than seems desirable for comparisons with the pole, in the case of the plates with four exposures.

Several comparison scales are in use, in order to suit the various qualities of definition on different plates, that one being chosen which best matches the images on the particular plate to be measured. The scales are cut out of negatives containing fourteen exposures on a selected star, of durations so calculated as to give a magnitude interval between each exposure of about $0^{m}\cdot25$; the actual intervals are, however, determined anew for each plate measured. The star to be measured and the comparison scale are viewed simultaneously in the eyepiece of a comparator by an arrangement of reflecting prisms. Estimates are made to one-tenth of the intervals on the comparison scale, and each plate is measured twice over by different people; a region 40′ square is measured on each plate.

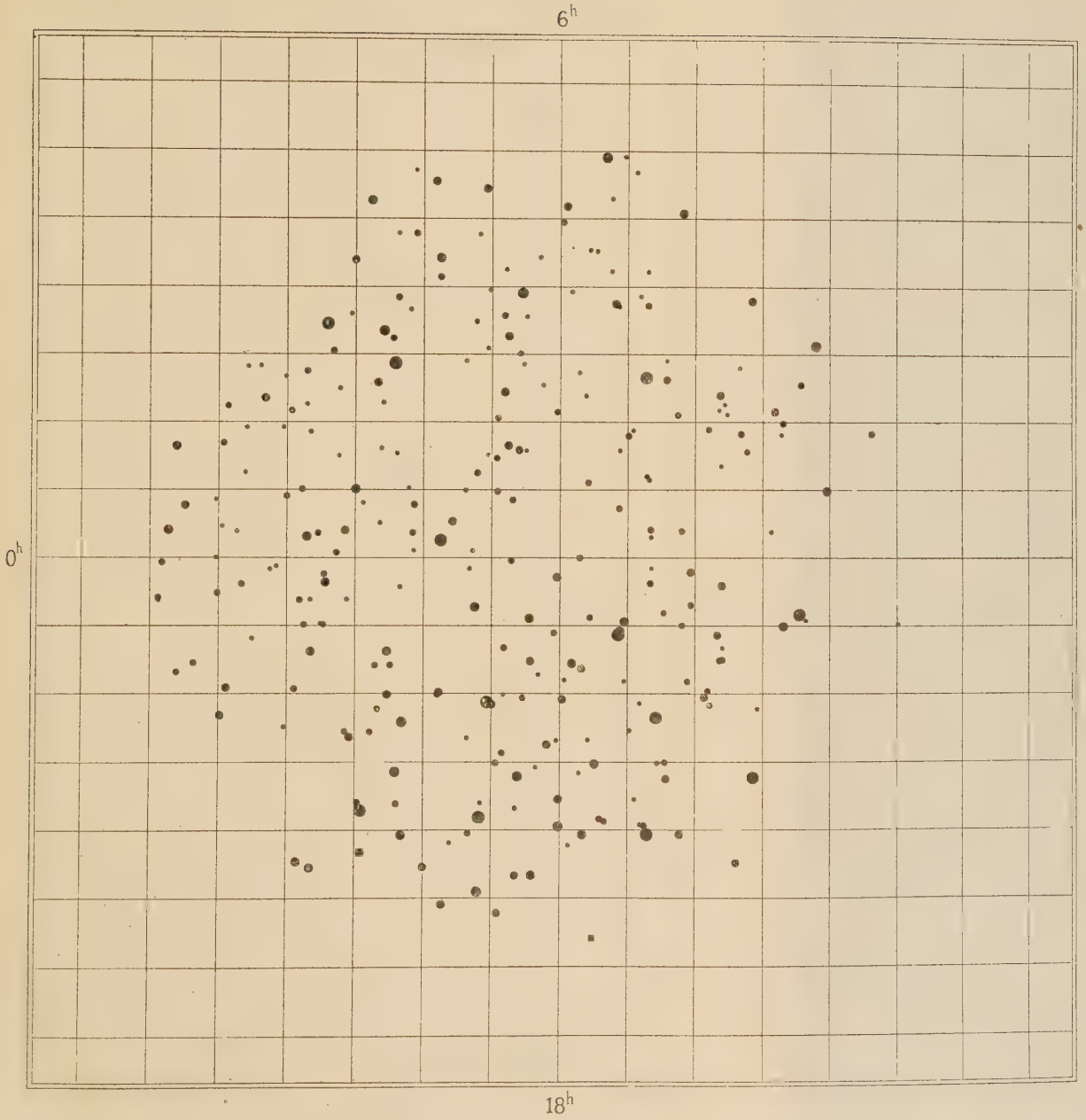
4. The comparison scale is standardised on every plate measured, by means of the stars of known magnitude on the pole field. This rests ultimately on Pickering's determination of the

photographic magnitudes of 17 stars (down to magnitude 14·19) in his polar sequence, but in order to diminish the effect of accidental errors in the measurement of these stars, the magnitudes of *all* the pole stars in the field have first been determined. For this purpose a number of the short-exposure plates were used, and, in addition, six plates each with an hour's exposure (and in one case with duplicate hour exposures). The latter were taken partly to test the diffraction gratings and partly to extend the magnitude determinations on the pole field to below 15^m, as well as to improve the magnitudes of stars near the limit of faintness of the short-exposure plates. The polar region is photographed on these plates at all hour angles, and in varying positions on the plate, so that the total region measured is a circle of about 50′ diameter; some stars near the edge and some faint stars occur only on a few out of the twenty plates used for the purpose. On account of the varying positions of the stars on the plates and the excellence of definition over the region measured, no correction to the magnitude depending on position on the plate has been applied.

5. The magnitudes thus determined of 262 stars are given in the following list. The values derived from the long and short exposure plates are shown separately, together with the number of plates on which the star is measured, in each case. The first column of the list gives the number of the star in *Harvard Annals*, xlviii., No. 1, Table VI. (Catalogue of stars near the North Pole). The second and third columns give the co-ordinates of the stars as measured from the pole (epoch 1900). The unit is 5′ of arc, and the x and y co-ordinates increase in the directions from the pole to 0^h and 6^h respectively. These co-ordinates are directly comparable with those given in the "Catalogue of 2212 Stars within 3° of the Pole," *Greenwich Astrographic Catalogue*, vol. iii Columns four and five refer to the long-exposure plates, and the sixth and seventh columns to the short-exposure plates. The adopted mean value is given in column eight, and in the ninth column the Harvard magnitudes (H. C., No. 170), on which the whole work is based, are inserted.

The interpolation between the magnitudes of the standard stars of the sequence, to arrive at the magnitudes of the other stars, was made graphically. From the residuals it appears that the probable error of an estimate on a single long-exposure plate is $\pm 0^m{\cdot}077$; this value applies to the faintest stars (below 15^m) as well as to those of magnitude 12, 13, and 14. On the short-exposure plates, similarly, the probable error of the estimate from a single plate is $\pm 0^m{\cdot}080$. For the stars down to the 14th magnitude, the number of plates on which each star occurs is numerous, from 6 to 15 or so; the stars $14^m{\cdot}0$ to $15^m{\cdot}0$ have in the mean 5 determinations, while those still fainter have from 1 to 3 or more determinations.

For the convenience of other observers the stars in this list have also been set out on the chart, Plate 1, the images being graded according to magnitude. The chart is probably complete to magnitude 15·0 at least, except possibly near its edge.



15-16. 14-15. 13-14. 12-13. 11-12. >11.

Chart of Stars within 25′ of the North Pole. Epoch 1900·0. One interval=5′.

The position of the Pole changes about one interval in 15 years in the direction of 0h.

Magnitudes of 262 Stars within 25′ of North Pole.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (Mean).	Harvard Mag.
31	−5·02	−0·99	15·72	1	...	...	15·72	...
41	−4·62	+1·83	14·95	1	...	...	14·95	...
57	−3·95	+0·98	12·60	5	12·60	7	12·60	...
62	−3·79	+3·12	...	...	11·28	4	11·28	...
70	−3·64	−0·93	15·45	1	...	...	15·45	...
74	−3·57	+2·55	14·72	3	...	...	14·72	...
75	−3·55	−0·84	...	...	10·90	6	10·90	...
85	−3·31	−1·01	12·26	3	12·32	8	12·29	...
86	−3·31	+1·98	14·53	6	...	...	14·53	...
88	−3·28	+1·81	15·00	2	...	...	15 00	...
90	−3·19	+2·16	13·31	6	13·33	7	13·32	...
94	−3·14	+0·38	15·40	2	...	...	15·40	...
100	−2·93	−2·22	15·53	1	...	...	15·53	...
101	−2·87	−3·24	...	...	10·26*	2	10·26	10·17
103	−2·86	+3·79	...	...	13·08	6	13·08	...
105	−2·78	+1·56	14·72	6	...	...	14·72	...
111	−2·69	+1·82	14·63	4	...	...	14·63	...
112	−2·67	+2·80	15·67	1	...	...	15·67	...
115	−2·61	−4·49	...	...	13·12	2	13·12	..
122	−2·48	+2·11	15·39	3	...	...	15·39	...
123	−2·45	+2·26	15·69	1	...	...	15·69	...
125	−2·41	−1·50	14·63	6	...	...	14·63	...
126	−2·41	−1·33	15·29	3	...	...	15·29	...
127	−2·40	−0·41	13·72	6	13·68	7	13·70	...
128	−2·40	+1·35	15·27	4	...	...	15·27	...
130	−2·39	−1·51	14·42	6	...	...	14·42	...
131	−2·37	+2·39	13·90	6	...	...	13·90	...
132	−2·36	+2·17	15·40	1	...	...	15·40	...
134	−2·34	−1·14	13·79	6	...	...	13·79	...
139	−2·23	−2·17	14·82	3	...	..	14·82	...
140	−2·20	+1·89	14·66	6	...	...	14·66	...
142	−2·19	−1·97	14·04	6	...	...	14·04	...
146	−2·14	−2·06	13·82	6	...	...	13·82	...
151	−1·95	−0·22	13·18	6	13·20	12	13·19	...
152	−1·94	−0·69	14·11	6	...	...	14·11	...
154	−1·90	−1·82	14·66	5	...	...	14·66	...
157	−1·83	+5·08	...	...	12·52	3	12·52	...

* From diffraction images with grating II. (see next paper).

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (mean).	Harvard Mag.
158	−1·81	−1·00	14·96	3	...	...	14·96	...
159	−1·81	+0·39	14·96	3	...	...	14·96	...
162	−1·78	−4·07	13·19	1	13·26	4	13·23	...
163	−1·75	+2·10	14·97	3	...	...	14·97	...
171	−1·59	+2·90	15·43	1	...	...	15·43	...
172	−1·59	+2·62	13·12	6	13·16	10	13·14	...
174	−1·58	−3·26	13·12	4	13·13	9	13·13	...
175	−1·56	−3·01	14·67	3	...	...	14·67	...
176	−1·54	−0·81	14·07	5	...	...	14·07	...
182	−1·44	−3·03	15·47	3	...	...	15·47	...
184	−1·43	−2·36	...	...	10·54	8	10·54	...
188	−1·35	−0·16	15·62	2	...	...	15·62	...
189	−1·34	+0·30	15·50	4	...	...	15·50	15·52
190	−1·34	+0·41	14·93	5	...	...	14·93	14·97
191	−1·34	−0·38	14·12	6	...	...	14·12	14·06
192	−1·32	+1·15	15·11	4	...	...	15·11	...
193	1·31	+3·72	14·82	5	...	...	14·82	...
194	−1·30	+4·21	15·65	1	...	...	15·65	...
195	−1·29	−4·07	...	...	10·94	7	10·94	...
196	−1·29	+1·20	15·04	4	...	..	15·04	...
197	−1·28	+2·65	...	...	10·95	10	10·95	10·94
199	-1·25	−3·94	14·99	1	...	...	14·99	...
201	−1·20	+3·86	15·14	4	...	...	15·14	...
204	−1·19	−3·92	15·40	1	...	...	15·40	...
205	−1·18	−2·14	15·44	2	...	...	15·44	...
206	−1·14	+5·69	15·29	1	...	...	15·29	...
209	1·10	−3·55	15·10	2	...	...	15·10	...
210	-1·09	+1·87	15·55	2	...	...	15·55	...
214	−1·03	−2·53	15·17	3	...	...	15·17	...
215	−1·02	+1·79	14·87	3	...	...	14·87	...
216	−0·97	+5·92	15·62	1	...	...	15·62	...
218	−0·96	−1·82	15·00	5	...	...	15·00	...
219	−0·95	−0·94	12·64	6	12·68	13	12·66	...
223	−0·89	+1·58	15·61	2	...	...	15·61	...
225	−0·88	−1·09	12·24	6	12·26	13	12·25	12·31
226	−0·87	+0·73	14·58	4	...	...	14·58	...
227	−0·87	−1·13	...	...	10·64	12	10·64	10·68
228	−0·87	+3·70	15·11	2	...	...	15·11	...
230	−0·83	+3·74	13·69	5	13·76	1	13·73	...

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (mean).	Harvard Mag.
233	−·078	+5·30	15·53	1	..	...	15·53	...
234	−0·77	+4·23	15·33	5	...	...	15·33	...
237	−0·69	+5·90	11·89	1	...	..	11·89	...
239	−0·66	−3·87	14·93	3	...	...	14·93	...
243	−0·59	−3·84	14·02	4	...	...	14·02	...
246	−0·56	+4·52	15·00	5	...	...	15·00	...
248	−0·51	−3·03	12·87	6	12·86	12	12·86	12·82
249	−0·48	−5·59	14·98	1	...	...	14·98	...
252	−0·45	+4·54	15·18	2	...	...	15·18	...
254	−0·44	−0·88	14·94	6	...	...	14·94	14·91
256	−0·43	+1·10	14·11	4	...	...	14·11	...
258	−0·41	−2·67	15·49	2	...	...	15·49	...
260	−0·39	+2·39	15·71	1	...	...	15·71	...
262	−0·33	−4·07	12·72	4	12·70	7	12·71	..
263	−0·32	−1·63	13·73	6	13·77	6	13·75	13·73
264	−0·29	+2·73	15·26	5	...	...	15·26	...
265	−0·29	0·00	14·33	6	..	...	14·33	14·33
266	−0·28	−3·17	15·66	1	...	...	15·66	...
268	−0·18	−1·55	12·34	6	12·30	13	12·32	12·28
269	−0·18	+3·92	15·44	1	...	...	15·44	...
273	−0·13	−4·22	15·14	3	...	...	15·14	...
275	−0·10	+5·18	13·98	2	...	..	13·98	...
277	−0·07	−1·80	15·09	5	...	...	15·09	...
278	−0·06	+4·96	14·89	2	...	...	14·89	...
279	−0·02	−2·09	13·55	6	13·51	6	13·53	13·50
281	+0·02	−3·55	13·85	6	13·84	4	13·85	...
282	+0·02	−3·95	...	...	11·47	7	11·47	11·49
283	+0·04	+2·15	14·33	4	...	...	14·33	...
284	+0·04	−0·29	13·20	6	13·20	12	13·02	13·20
286	+0·05	−2·68	15·49	2	...	...	15·49	...
288	+0·08	−1·10	14·15	6	...	...	14·15	14·19
292	+0·19	−2·75	12·99	6	13·05	12	13·02	13·05
293	+0·24	+2·55	15·14	5	...	...	15·14	...
294	+0·29	+4·44	15·10	3	...	...	15·10	...
295	+0·31	−1·72	15·26	3	...	...	15·26	...
297	+0·34	−3·07	15·40	2	...	...	15·40	...
300	+0·43	−1·52	13·22	6	13·15	11	13·19	...
302	+0·45	−0·89	12·60	6	12·62	12	12·61	12·59
303	+0·48	+3·56	15·27	5	...	...	15·27	...

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	*x*.	*y*.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (Mean).	Harvard Mag.
304	+0·49	+1·59	15·43	1	...	...	15·43	
306	+0·52	+2·86	14·90	6	...	...	14·90	...
309	+0·54	+3·90	...	...	11·03	12	11·03	...
311	+0·55	−2·06	14·70	4	...	...	14·70	...
312	+0·58	+3·02	15·42	2	...	...	15·42	...
313	+0·58	−4·66	12·90	1	13·06	5	12·98	...
319	+0·60	+1·58	13·95	3	...	...	13·95	...
320	+0·62	−3·22	11·95	4	11·99	12	11·97	11·92
321	+0·65	−4·67	13·84	1	...	...	13·84	...
322	+0·66	−3·69	15·80	2	...	...	15·80	...
323	+0·69	+0·85	14·87	4	...	...	14·87	14·88
324	+0·71	−0·04	14·56	6	...	...	14·56	14·55
327	+0·75	+3·27	13·99	6	...	...	13·99	...
329	+0·76	+1·65	13·88	6	13·82	4	13·85	...
331	+0·78	+4·26	15·63	1	...	...	15·63	...
332	+0·80	+2·44	13·85	6	...	...	13·85	...
333	+0·81	+3·58	14·32	6	...	...	14·32	...
334	+0·82	−1·32	14·90	6	...	...	14·90	...
335	+0·83	−2·01	15·76	2	...	...	15·76	...
336	+0·85	−2·87	14·50	6	...	...	14·50	...
338	+0·90	+2·06	14·27	6	...	...	14·27	...
339	+0·91	+0·98	14·99	5	...	...	14·99	15·09
340	+0·92	−5·23	...	...	13·50	3	13·50	...
341	+0·92	+1·46	14·80	5	...	...	14·80	...
342	+0·94	−3·02	14·62	6	...	...	14·62	...
345	+1·00	−2·15	13·10	6	13·16	11	13·13	...
346	+1·02	+3·95	15·38	2	...	...	15·38	...
347	+1·05	+1·52	15·50	2	...	...	15·50	...
348	+1·06	+3·09	15·10	5	...	...	15·10	...
349	+1·07	+5·45	13·21	3	13·11	3	13·16	...
350	+1·07	−2·13	...	...	10·64	11	10·64	...
354	+1·16	−3·60	15·17	4	..	...	15·17	...
355	+1·17	+4·78	15·34	2	...	...	15·34	...
356	+1·18	−3·82	...	...	10·42	9	10·42	...
358	+1·20	+1·26	14·99	6	...	...	14·99	...
359	+1·21	−4·91	...	...	10·98	4	10·98	...
360	+1·21	+3·49	15·68	2	...	...	15·68	...
362	+1·25	−0·73	12·61	6	12·59	13	12·60	...
363	+1·28	+0·10	15·63	1	...	...	15·63	15·61

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (Mean).	Harvard Mag.
365	+1·32	−0·16	15·02	4	...	...	15·02	14·97
366	+1·34	−4·06	14·84	3	...	...	14·84	...
368	+1·36	−2·65	15·32	3	...	...	15·32	...
369	+1·37	+2·90	15·49	3	...	...	15·49	...
370	+1·39	+1·00	15·69	2	...	...	15·69	...
378	+1·57	+0·54	13·51	3	13·41	8	13·46	...
380	+1·61	−4·19	15·80	2	...	...	15·80	...
385	+1·73	−5·09	...	...	13·28	2	13·28	...
386	+1·74	+0·26	...	...	10·62	11	10·62	10·52
387	+1·75	+4·42	13·02	5	12·95	6	12·98	...
388	+1·76	+4·14	14·98	2	...	...	14·98	...
389	+1·78	−1·98	13·11	6	13·09	13	13·10	...
391	+1·79	−2·01	14·15	6	...	...	14·15	...
393	+1·82	+5·55	13·93	1	...	...	13·93	...
398	+2·01	−4·56	13·06	2	13·15	5	13·10	...
399	+2·10	+4·78	14·14	3	...	...	14·14	...
400	+2·11	+5·72	15·01	1	...	...	15·01	...
401	+2·13	+0·78	14·10	6	...	...	14·10	...
402	+2·14	+0·10	15·43	3	...	...	15·43	...
403	+2·16	+0·37	14·22	6	...	...	14·22	...
404	+2·19	+3·67	15·48	2	...	...	15·48	...
405	+2·21	+1·02	15·59	1	...	...	15·59	...
408	+2·32	−2·42	...	...	11·26	12	11·26	11·26
409	+2·33	−4·07	12·04	1	12·00	6	12·02	...
410	+2·34	−0·43	15·07	4	...	...	15·07	...
412	+2·36	+4·79	15·03	1	...	...	15·03	...
413	+2·37	+3·84	14·84	4	...	...	14·84	...
417	+2·39	+1·54	15·30	5	...	...	15·30	...
418	+2·40	−3·61	14·55	3	...	...	14·55	...
419	+2·42	3·15	...	...	11·12	8	11·12	...
420	+2·43	+2·87	8·71	6	8·86*	23	8·83	8·89
421	+2·45	+3·23	14·99	3	...	...	14·99	...
424	+2·48	−1·58	14·13	6	...	...	14·13	...
426	+2·53	−2·02	13·36	6	13·37	8	13·36	...
427	+2·53	−1·37	12·68	6	12·63	13	12·65	...
430	+2·58	+3·34	...	...	11·57	13	11·57	...
431	+2·59	+2·29	15·46	2	...	...	15·46	...

* From diffraction images with grating II. (see next paper).

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (Mean).	Harvard Mag.
433	+2·62	+1·62	15·70	1	...	...	15·70	...
435	+2·64	+0·52	15·52	1	...	...	15·52	...
436	+2·67	+2·58	13·12	6	13·19	12	13·16	...
437	+2·67	−2·23	14·74	6	...	...	14·74	...
439	+2·71	−1·57	14·11	5	...	...	14·11	...
441	+2·76	+5·27	...	...	12·28	6	12·28	...
442	+2·79	−2·56	14·23	6	...	...	14·23	...
444	+2·89	+0·81	15·49	1	...	...	15·49	...
446	+2·93	−4·32	...	...	13·04	6	13·04	...
447	+2·93	−3·72	...	...	10·69	8	10·69	...
451	+3·00	+1·01	13·01	6	12·88	12	12·95	...
452	+3·00	+4·39	13·94	3	...	...	13·94	...
455	+3·06	+3·60	15·37	1	...	...	15·37	...
456	+3·08	−2·63	13·35	3	13·33	9	13·34	...
458	+3·12	−0·60	15·30	1	...	...	15·30	...
459	+3·15	+0·40	13·93	6	...	...	13·93	...
460	+3·15	−2·56	14·50	2	...	...	14·50	...
462	+3·23	+2·49	15·40	1	...	...	15·40	...
464	+3·23	+1·50	15·22	3	...	...	15·22	...
468	+3·28	+0·08	14·95	6	...	...	14·95	...
472	+3·33	+3·04	14·96	2	...	...	14·96	...
477	+3·41	+3·44	...	...	10·66	7	10·66	...
479	+3·44	−0·35	12·85	6	12·93	13	12·89	...
480	+3·46	−0·24	14·33	6	...	...	14·33	...
481	+3·47	−0·98	14·85	6	...	...	14·85	...
482	+3·51	−0·97	15·60	1	...	...	15·60	...
485	+3·54	+0·36	14·85	4	...	...	14·85	...
488	+3·65	+1·86	15·46	2	...	...	15·46	...
489	+3·65	−1·37	12·82	6	12·82	13	12·82	...
490	+3·65	−4·56	...	...	12·56	2	12·56	...
491	+3·66	−0·61	15·29	6	...	...	15·29	...
494	+3·70	+0·31	12·10	5	12·14	13	12·12	...
495	+3·71	+2·26	15·55	1	...	...	15·55	...
496	+3·71	+2·75	14·81	3	...	...	14·81	...
498	+3·75	−0·99	14·82	6	...	...	14·82	...
499	+3·76	−0·96	14·70	6	...	...	14·70	...
502	+3·78	+1·01	14·94	3	...	...	14·94	...
504	+3·81	−0·62	14·63	6	...	...	14·63	...
506	+3·86	−4·46	...	...	12·56	2	12·56	...

Magnitudes of 262 Stars within 25′ of North Pole—Continued.

Harvard No. (48).	x.	y.	Mag.	No. of Plates.	Mag.	No. of Plates.	Mag. (Mean).	Harvard Mag.
507	+3·90	−1·93	14·67	4	...	...	14·67	...
509	+3·93	+2·17	14·92	3	...	...	14·92	...
511	+4·00	+0·91	14·98	4	...	...	14·98	...
512	+4·02	+2·67	15·01	3	...	...	15·01	...
513	+4·04	−2·49	15·04	1	...	...	15·04	...
514	+4·06	+1·93	15·50	2	...	...	15·50	...
515	+4·16	−0·12	15·04	4	...	...	15·04	...
516	+4·24	−0·17	15·18	3	...	...	15·18	...
522	+4·32	+2·34	13·91	4	...	...	13·91	...
526	+4·39	+2·83	15·40	1	...	...	15·40	...
532	+4·50	−1·18	15·17	1	...	...	15·17	...
535	+4·57	+2·81	15·00	1	...	...	15·00	...
536	+4·59	+1·92	15·29	1	...	...	15·29	...
540	+4·61	+1·26	15·01	5	...	...	15·01	...
543	+4·66	−0·39	14·26	4	...	...	14·26	...
545	+4·73	+0·39	15·63	2	...	...	15·63	...
551	+4·87	+2·23	14·98	2	...	...	14·98	...
554	+4·89	−1·91	13·51	1	13·55	5	13·53	...
556	+4·92	+1·68	14·01	2	...	...	14·01	...
557	+4·94	+0·47	15·61	2	...	...	15·61	...
561	+4·98	−2·32	...	...	13·60	3	13·60	...
562	+5·01	−0·52	14·82	1	...	...	14·82	...
564	+5·03	+0·86	15·46	2	...	...	15·46	...
565	+5·04	0·00	15·66	1	...	...	15·66	...
572	+5·36	−1·54	14·88	1	...	...	14·88	...
574	+5·48	+0·76	...	...	13·12	6	13·12	...
576	+5·62	−1·68	14·54	1	...	...	14·54	...
577	+5·62	+1·64	12·09	2	11·96	5	12·03	...
579	+5·73	+0·41	...	...	12 99	7	12·99	...
582	+5·82	−0·07	...	...	14·03	2	14·03	...
583	+5·88	−0·59	14·87	1	...	...	14·87	...
584	+5·88	−0·62	15·38	1	...	...	15·38	...

On the Application of Parallel Wire Diffraction Gratings to Photographic Photometry. By S. Chapman, B.A., D.Sc., and P. J. Melotte.

(*Communicated by the Astronomer Royal.*)

1. The use of parallel wire diffraction gratings in photographic photometry is to enable magnitude *intervals* to be measured on a plate. Schwarzschild, Hertzsprung, and others have previously used diffraction gratings for this purpose (*A.N.*, vol. clxxxvi., No. 4452). The grating placed in front of the object-glass produces (by diffraction) auxiliary images on each side of the ordinary image of a star, diverting a definite proportion of the light into each image; the proportion is a constant, depending only on the dimensions of the grating; thus there is a constant magnitude interval between the central image and the diffraction images on either side of it.* There is this magnitude interval also, therefore, between the given star and any other star whose *central* image is equal in size and greyness to the diffraction images of the former star. By utilisation of this fact a simple method is obtained of determining the magnitudes of all the stars on a plate, apart from a constant zero correction, when the magnitude interval of the grating is known; and if the magnitudes of some of the brighter stars on the plate are also known, the zero correction to all the magnitudes is at once determinable.†

2. The special advantage of a diffraction grating for measuring magnitude intervals is that the auxiliary images are produced at the same time as the central images, a definite amount of the light always going to each. The magnitude interval is therefore independent of the constancy of the transparency of the atmosphere during the exposure of the plate, an advantage not shared by any method which depends on an auxiliary exposure with the objective partly screened off or obscured in some way.

A theoretical disadvantage of the diffraction method, which can, however, be rendered of no practical importance by suitably designing the grating, lies in the fact that the auxiliary images are really small spectra, light of different wave-lengths being dispersed in slightly different directions, so that the diffraction images are a trifle elongated, and consequently not strictly comparable with the round central images of fainter stars. This will be considered in more detail later, but for the present it may be stated that sensibly round images are obtained with the second of the two gratings

* Theoretically there is an infinite series of images on each side of the central one, but most of these are much too faint to appear. Generally only the brighter stars on a plate show any diffraction images at all, and rarely are more than two seen, with the gratings and exposure times adopted at Greenwich.

† See the previous paper on "Photographic Magnitudes."

discussed in this paper, the one first made having shown a slight but perceptible elongation of the images. Sensibly round images are obtainable on account of the fact that the photographic image of a star is not a point, but a disc whose diameter is fairly large compared with the elongation produced by the varying dispersion of the different parts of the spectrum.

A more serious disadvantage of the grating is that it diminishes the effective aperture of the telescope, owing to the light stopped up by the opaque wires or strips. This necessitates an increase in the time of exposure in order to obtain images of stars as faint as would be obtained without the grating. This drawback is real and unavoidable, but the advantages of the method as a whole seem to outweigh any objection on this score.

3. In designing a grating the following points must be considered, therefore:—

(*a*) *The Dispersion.*—If the grating consists of parallel wires or strips of width d separated by spaces of width a, while f is the focal length of the telescope (all measured in centimetres), the distance D between the central image and the first diffraction image produced by the light of wave-length λ cm. will be given in linear measure by

$$D = \frac{f\lambda}{a+d}.$$

If the part of the spectrum which acts on the photographic plate ranges from λ_1 to λ_2, the elongation of the image will be $\frac{f(\lambda_1 - \lambda_2)}{a+d}$. To diminish this elongation, for a given type of star, we must diminish $\frac{f}{a+d}$, *i.e.* we must choose such a pitch $(a+d)$ for the grating as will make the central and diffraction images as close together as possible. The limit of nearness will depend on the size of the images given by the telescope, as it is important that the two images shall be quite separate and distinct.

In the first grating referred to in this paper the distance between the central and diffraction images was 0·062 cm., which was unnecessarily large, and therefore, to reduce the slight elongation of the images which this separation produced, the second grating was designed to give a separation of 0·045 cm.

The focal length of the refractor is 6·837 metres, and the pitch of the grating is in Case I. 0·5000 cm., and in Case II. 0·7000 cm.; the mean wave-length of the effective light of the stars is about 4500 tenth-metres, agreeing with the above values of the "separation."

(*b*) The separation of the images depends only on the "pitch" or number of elements in the grating. The *absorption* of the grating is independent of the pitch, being determined by the ratio of the blank space (a) to the width of the wire or strip (d). The absorption of the grating (*i.e.* the difference in magnitude between the central image, using the diffraction grating, and the

image obtained with an equal exposure in the absence of the grating) is expressed in magnitudes by

$$5 \log_{10} \frac{a+d}{a}.$$

Expressed otherwise, to get the same image in the two cases the exposure must be increased, when the grating is used, in the ratio $\left(\frac{a+d}{a}\right)^2$.

In the case of grating I. the value of $\left(\frac{a}{a+d}\right)$ was 0·8614 and in grating II. was 0·7547, giving a loss of magnitude of 0·325 and 0·610, or requiring an increase of exposure time in the ratio 1·35 and 1·76 respectively. It will be seen from the formulæ that the effective loss in magnitude varies as the square of the ratio of the original aperture to the aperture as diminished by the opaque strips or wires. It is advisable, therefore, to keep the opaque elements of the grating as narrow as possible consistent with the fulfilment of the other requirements.

(*c*) *The Magnitude Interval.*—The interval of magnitude between the central and first diffraction image is given by the formula

$$2{\cdot}5 \log_{10} \frac{\left(\frac{a\pi}{a+d}\right)^2}{\sin^2\left(\frac{a\pi}{a+d}\right)} = 5\left(\log \frac{a\pi}{a+d} - \log \sin \frac{a\pi}{a+d}\right).$$

Thus the magnitude interval, like the absorption, depends only on the ratio of the blank space to the covered part of the grating. The following table (No. I.) may be useful in determining the value of $\frac{a}{a+d}$ to give a required magnitude interval. The first column gives the ratio of the blank space to the pitch, the second the ratio of covered space to the pitch, the third the corresponding magnitude interval between the diffraction and central images, and the fourth the loss of magnitude of the central image due to the grating.

The diameters of the wires (a) in gratings I. and II. are (from the mean of a large number of measures) 0·0693 cm. and 0·1717 cm., and the pitch ($a+d$) 0·5000 cm. and 0·7000 cm. respectively; hence it is easily seen that the calculated magnitude intervals are 4·036 and 2·660 respectively.

4. As to the best choice of a magnitude interval for a grating, the following considerations must be taken into account. For ordinary photographic purposes, in which it is required to extend the magnitude scale from a number of stars whose magnitudes are known, to other faint stars, it is generally inadvisable to have a greater interval than three magnitudes, since the greater the

magnitude interval the fewer of course are the stars showing diffraction images.

Table I.

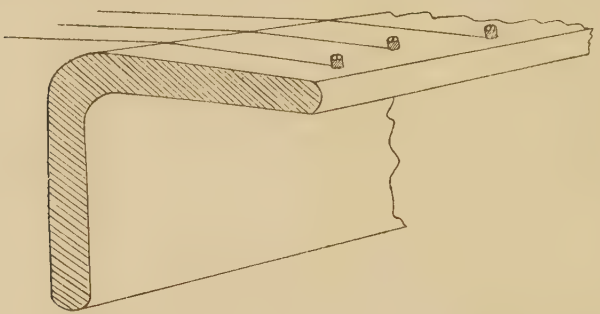
$\frac{a}{a+d}$	$\frac{d}{a+d}$	Magnitude Interval.	Absorption in Magnitudes.	$\frac{a}{a+d}$	$\frac{d}{a+d}$	Magnitude Interval.	Absorption in Magnitudes.
·950	·050	6·402	·111	·775	·225	2·870	·554
·925	·075	5·476	·170	·750	·250	2·614	·624
·900	·100	4·806	·229	·725	·275	2·382	·698
·890	·110	4·583	·253	·700	·300	2·171	·774
·880	·120	4·378	·278	·675	·325	1·978	·854
·870	·130	4·188	·303	·650	·350	1·800	·936
·860	·140	4·012	·328	·625	·375	1·638	1·020
·850	·150	3·848	·353	·600	·400	1·486	1·109
·840	·160	3·694	·378	·575	·425	1·345	1·202
·830	·170	3·547	·404	·550	·450	1·214	1·298
·820	·180	3·410	·431	·525	·475	1·093	1·399
·810	·190	3·279	·458	·500	·500	0·980	1·505
·800	·200	3·155	·484				

If m is the magnitude of the faintest stars shown on the plates taken, and μ is the magnitude interval of the grating, only stars brighter than those of magnitude $m-\mu$ will show diffraction images. Thus μ must be chosen so that within the field of good definition of the plates, and with the exposures allowable, a sufficient number (say 10 or 12) of stars brighter than $m-\mu$ can be found. On the other hand, if the exposure time is a consideration, μ should not be much less than 2, otherwise the loss of light with the grating will be excessive; and in any case the grating method is most profitable and accurate when the interval μ is nearly as large as the range of magnitude corresponding to the range of measurable diameters or greyness (this is rarely more than four magnitudes), otherwise the extension of the magnitude scale on the plates involves more than one "step down" by the grating. If, however, the diffraction grating is to be used to check the magnitude scale of a given sequence, such as the North Polar Sequence, a larger magnitude interval may be desirable. In this case, if great accuracy is required, the magnitude interval of the grating should be determined by photometric experiments in the laboratory, as the ratio $d/(a+d)$ needs to be determined very accurately to get good determination of μ when this much exceeds 4.

The two gratings used in the present investigation have, as we have already seen, magnitude intervals of amount 4·04 and 2·66; the first one was made partly for the purpose of checking the magnitude scale of the North Polar Sequence, or at any rate to check the homogeneity of the scale in different parts, and for this it has proved not unsuitable. But it was not very well adapted for the purpose of extending the magnitude scale of the long-

exposure plates on the Franklin-Adams' regions (see the paper on "Photographic Magnitudes of 262 Stars," §§ 1, 2); the short-exposure plates there referred to give the magnitudes of stars down to $13^{m}{\cdot}5$, while the long-exposure plates rarely go down below $15^{m}{\cdot}5$; only stars brighter than $11{\cdot}5$, therefore, will give diffraction images with grating I. for the determination of the magnitudes from $13{\cdot}5$ to $15{\cdot}5$, and of these there are about a dozen within the area (40′ square) measured on the plates, in an average part of the sky. This is just about sufficient, but leaves no margin in case the sky is less transparent or the star density smaller than usual, and grating II. with the magnitude interval 2·66 has therefore been adopted for the ordinary photometric work.

5. The construction of the gratings is very similar. As they were originally made for use with the 30-inch reflector, their aperture is 30 inches in diameter. Each consists of a large metal cylindrical ring with a flange turned over at rather more than a right angle to the cylindrical portion. The metal is $\frac{3}{8}$ inch in



thickness, the ring being $2\frac{1}{4}$ inches high and the flange $2\frac{1}{2}$ inches wide. The shape can be seen from the accompanying sketch. Owing to the large size, the metal used (upon the advice of Messrs. Troughton & Simms, who constructed the two gratings) was "vanalium," an alloy of aluminium combining lightness with considerable strength; the weight of each grating with wires is about 16 lbs. The flange was bevelled in order that the wires, which lie in V grooves accurately marked out on a measuring machine, may be kept taut and in their proper positions by tension applied by capstan screws near the rim of the flange. The tension applied to each wire is equivalent to about 8 or 10 lbs. weight; the bevelling of the flange gives the ring greater strength to bear the considerable distorting force thus brought to bear upon it. The capstan screws are placed in zigzag fashion, in order to clear one another. In the case of grating I. the wires cross the ring three times between each pair of screws, passing twice through holes bored through the ring opposite the grooves. In the case of grating I. fine copper wire was used, while in grating II. aluminium wire was adopted, as being softer and more easily kept straight

without excessive tension than copper wire would be; a soft and non-rusting metal is of course necessary for the purpose. In each case the wires were finely coated with a dull black paint in order to avoid reflection effects of any kind. The gratings were intended to be held in the reflector tube by sliding friction only, the cylindrical part fitting inside the tube. When the photometric work was transferred to the refractor, which has a smaller aperture, three lugs were screwed on to the inside of the cylindrical part of the ring, and these lugs slide into suitable tight-fitting grooves in castings which are attached to the telescope tube. The gratings are always placed in front of the object-glass in the same position, in such wise as to cause the line joining the diffraction images to lie in the R.A. direction on the plate.

6. The results obtained by the photographs of the region of the North Polar Sequence are given in Tables II. to IV. The magnitudes of all the stars on the Harvard photographic scale being known (see previous paper), the measures are for the present purpose regarded as determinations of the magnitude interval of the gratings in terms of the Harvard scale.

TABLE II.

Magnitude Interval of Grating I.

No.	Mag.	4853.	4859.	4900.	4901.	4912.	Mean of each Star.
420	8·89	3·92	4·06	3·97	3·92	3·99	$3·97_5$
356	10·42	...	...	...	4·44	4·32	$4·38_2$
184	10·54	4·15	4·12	4·04	4·32	4·25	$4·18_5$
386	10·62	4·08	4·02	4·42	4·18	4·15	$4·17_5$
227	10·64	4·04	4·09	4·26	4·29	4·16	$4·17_5$
350	10·64	4·19	3·97	4·11	4·43	4·23	$4·19_5$
477	10·66	4·15	4·10	3·86	4·10	...	$4·05_4$
75	10·90	...	...	...	4·50	4·48	$4·49_2$
195	10·94	4·15	...	...	...	...	$4·15_1$
197	10·95	4·01	4·47	4·48	4·27	4·25	$4·30_5$
309	11·03	4·23	4·26	4·28	4·13	4·35	$4·25_5$
419	11·12	...	...	...	4·29	4·15	$4·22_2$
408	11·26	4·18	4·29	3·99	4·34	4·27	$4·21_5$
282	11·47	...	...	4·14	...	4·29	$4·17_2$
430	11·57	4·14	...	...	4·14	4·20	$4·16_3$
...	...	4·12	4·16	4·16	4·26	4·24	...

Mean of 56 determinations = $4^m·19$.
Probable error of a single determination = $\pm 0^m·104$.

Table II. gives the results from five pole plates with 60 minutes exposure, one of which (No. 4901) had two such exposures. The

intervals as determined from each star on each plate are given separately; each plate showed about 11 stars with diffraction images, in the region measured. Each row refers to a separate star (the stars being arranged in order of magnitude), and each column to a separate plate. In the first and second columns are the number of the star and its magnitude from the list in the previous paper on "Photographic Magnitudes." In the last column the interval is given as determined from each star separately, and in the last row as got from each plate separately. Apart from the first star, which stands out somewhat,* there appears to be no sign of a systematic change in the measured interval depending on magnitude. The mean determinations from each plate separately agree very well, giving a mean value 4·19 from 56 separate measures of stars. The probable error of each individual measure is $\pm 0^m{\cdot}104$.

Besides this value from the long-exposure plates, a number of measures have been made on the short-exposure plates on the polar region; these plates only show diffraction images for one star, No. 420, all the others being too faint. The following table gives the equivalent magnitude of the diffraction images of this star on fifteen short-exposure plates:—

TABLE III.

13·00	13·31	13·00	12·90	12·95
12·90	13·14	12·91	13·09	12·90
13·05	13·01	13·11	12·89	13·12

If the Harvard magnitude of this star (8·89, see previous note) is correct, the magnitude interval of the grating thus derived is $13{\cdot}02 - 8{\cdot}89 = 4{\cdot}13$, 13·02 being the mean of the above determinations. Combining this measure of the interval with the preceding, weighted according to the number of separate star images measured in each case, we get

Magnitude interval of grating I. $= 4^m{\cdot}18$.

In regard to grating II., the results are given in Table IV. in a manner similar to those for grating I. in Table II. Plate 4996 is a 60-minute exposure plate, while all the others are 6-minute exposure plates. As before, no systematic relation between the measured interval and the magnitude of the star is observable (the last six stars are measured on one plate only), while the mean values for each plate again agree very well. The mean value of the interval as determined from 98 images is $2^m{\cdot}71$, while the probable error of the measure from a *single* image is $\pm 0^m{\cdot}094$.

* If Professor Seares' correction to the Harvard magnitude of this star is correct (he gives it as 8·65 instead of 8·89), the discrepancy would disappear. See *Ap. J.*, xxxviii. p. 248, star No. 10 of Table II.

TABLE IV.

Magnitude Interval of Grating II.

No.	Mag.	Number of Plate. 4996.	5014.	5061.	4922.	4923.	4924.	4997.	5059.	5070.	5023.	Mean for each Star.
420	8·89	2·68	2·66	3·09	2·66	2·73	2·31	2·65	2·67	2·94	...	$2·71_9$
356	10·42	2·67	2·90	2·62	2·67	2·75	2·86	2·98	2·60	2·68	2·57	$2·73_{10}$
184	10·54	2·74	2·86	2·84	2·68	2·66	...	2·67	2·46	2·60	2·75	$2·70_9$
386	10·62	2·56	2·54	2·64	2·78	2·62	2·69	2·72	2·65	2·59	...	$2·64_9$
227	10·64	2·49	2·58	2·77	2·70	2·70	2·77	2·66	2·72	2·50	...	$2·65_9$
350	10·64	2·56	2·69	2·62	2·70	2·70	2·73	2·66	2·61	2·46	...	$2·62_8$
477	10·66	...	...	...	2·76	2·63	2·64	...	...	...	...	$2·67_3$
447	10·69	...	...	2·72	2·74	2·66	2·94	...	...	2·82	...	$2·78_5$
75	10·90	2·78	2·73	2·91	2·64	3·03	...	...	...	...	...	$2·82_5$
195	10·94	2·82	...	2·85	...	...	...	...	...	...	...	$2·83_2$
197	10·95	2·79	2·95	2·77	2·60	2·59	2·99	...	...	...	...	$2·81_6$
359	11·01	...	2·69	2·89	...	...	...	...	...	...	...	$2·79_2$
309	11·03	2·76	2·79	2·62	2·55	2·97	...	...	...	...	...	$2·74_5$
419	11·12	2·65	...	2·80	2·41	2·87	2·86	...	...	...	...	$2·72_5$
408	11·26	2·67	...	2·77	2·32	2·87	...	...	...	...	...	$2·66_4$
282	11·47	3·00	...	...	...	...	...	...	...	...	...	$3·00_1$
430	11·57	2·68	...	...	...	...	...	...	...	...	...	$2·68_1$
320	11·97	2·63	...	...	...	...	...	...	...	...	...	$2·63_1$
494	12·12	2·54	...	...	...	...	...	...	...	...	...	$2·54_1$
225	12·25	2·83	...	...	...	...	...	...	...	...	...	$2·83_1$
268	12·32	2·79	...	...	...	...	...	...	...	...	...	$2·79_1$
Mean for each plate		2·70	2·74	2·78	2·64	2·70	2·76	2·73	2·63	2·66	2·66	

Mean of 98 determinations = $2^m·71$.
Probable error of a single estimate = $\pm 0^m·094$.

CONCLUSIONS.

The fact that the intervals of the two gratings as determined from different stars show no dependence on the magnitude of the star indicates that, within the limits of error, the Harvard photographic scale is "homogeneous" (*i.e.* the light ratio corresponding to an interval of one magnitude is constant) from the 10th to the 15th magnitude (the case of star No. 420 is slightly uncertain). This result agrees, so far as it goes, with that got by Professor Seares at Mount Wilson in his work on the North Polar Sequence.*

The close accordance of the magnitude intervals of the two gratings as determined on a number of different plates indicates that the grating method is very suitable for measuring intervals of 3 or 4 magnitudes, in the manner here related, and is independent of variations in the atmospheric transparency during the exposure—an important point in the photometry of faint stars.

* *Ap. J.*, xxxviii pp. 241–267, 1913.

A comparison of the calculated and measured magnitude intervals of the grating gives the following results:—

	Grating I.	Grating II.
Calculated	2·66	4·04
Measured	2·72	4·18
Difference (calculated – measured) . .	−0·06	−0·14

The discrepancy is not large, and to this extent the investigation may be regarded as a confirmation of the accuracy of the light ratio of the magnitude scale (from 10 to 15·5) of the North Polar Sequence. But the difference cannot be attributed to accidental error either in the determination of the dimensions of the grating * or (as Table II. shows) in the measures of the plates. Nor, although the formulæ of this paper (which are adapted from Lord Rayleigh's article on the "Diffraction of Light" in the *Encyclopædia Britannica*) are arrived at on the assumption that the grating is composed of an infinite number of infinitely long parallel strips, does it seem possible that any error of measurable amount can arise on account of the finite boundary. Another conceivable source of error is a minute increase in the effective diameter of the wires due to moisture condensing upon them, but this would have the effect (see Table I.) of diminishing the measured magnitude interval, and so would increase the discrepancy.

Apparently there remain only two possible explanations: either the fact that the images are not truly circular produces a small systematic error in the comparison of the diffraction images with the central images of faint stars, or else there is a small percentage error in the light ratio of the Harvard photographic scale. The fact that the differences with the two gratings are of the same sign and nearly proportional to the respective magnitude intervals is accordant with the latter hypothesis. The images, especially with grating II., are perceptibly round, and it is therefore difficult to see how the former explanation can hold good: on the other hand, there is the Mount Wilson confirmation of the scale value of the Harvard Sequence over the interval (10·0 to 15·5) in question. For the present it would seem advisable to suspend judgment on the matter till further confirmation of one or other view is obtained. Meanwhile, as the Harvard scale is the adopted basis of the photometry work in progress, the grating interval determined from the plates will be used in reducing the further results from the gratings, a course which will render it possible to correct the whole mass of results in a uniform manner, should any small correction to the Harvard scale be generally agreed upon in the future.

* An error of 0·001 inch in the determination of the diameter of the wire of the grating would lead to errors of 0·03 and 0·08 respectively in the calculated intervals of gratings II. and I.; but such an error is quite out of the question. Two independent measures of the dimensions of grating I., several months apart, led to results for the interval differing by only ·006 mag.

Errata in Mr. Denning's paper, Vol. lxxiii. Page 670 (*Table*).

Through an accident in the press the figures for Oct. 25–27 have slipped into the wrong columns, and should read as follows:—

Oct.	25	26	1	
	26	17	1	
	27	42	4	90+16